MATH 233 LECTURE 39: PARAMETRIC SURFACES AND SURFACE INTEGRALS

- Recall the form of a parametric curve in space: $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. If we want to parametrize a surface, we just need to let x, y, z depend on two variables:

$$\vec{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}.$$

  We think of this as a function $\vec{r}: D \to \mathbb{R}^3$ defined on a region $D \subset \mathbb{R}^2$ (in the uv-plane). Its image $S = \vec{r}(D)$ is the parametrized surface.
- Instead of $\vec{r}'(t)$ we now have two partial derivatives $\vec{r}_u(u, v)$ and $\vec{r}_v(u, v)$ obtained by taking partials of x, y, z with respect to u, v. The surface area of S is given in terms of these:

$$A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA.$$

- The integral of a function $f(x, y, z)$ over S is given by

$$\iint_S f(x, y, z) dS := \iint_D f(\vec{r}(u, v)) \, \|\vec{r}_u \times \vec{r}_v\| \, dA.$$

  In particular, the surface area is just the integral of the function "1" over S.
- In the special case that S is the graph of a function $z = f(x, y)$, you can use $\vec{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + f(x, y)\hat{k}$, and then (short computation) $\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + f_x^2 + f_y^2}$.
- To write an equation for the tangent plane to a parametric surface S at a point $\vec{r}(u_0, v_0)$ on S, you need (besides that point) a normal vector. This is given by $\vec{n} = \vec{r}_u(u_0, v_0) \times \vec{r}_v(u_0, v_0)$.