

**MATH 233 LECTURE 5 (§§12.5-6):  
DISTANCE IN SPACE; QUADRIC SURFACES**

**Distance.** Having derived a formula for the distance from a point to plane in the last lecture, we now consider:

- Point to line: this can be done using (1-variable) calculus or the cross product. For the latter, if  $Q$  is a point on the line  $\ell$ ,  $\vec{v}$  its direction vector, and  $P$  any point in space, then

$$d(P, \ell) = \frac{\|\overrightarrow{QP} \times \vec{v}\|}{\|\vec{v}\|}.$$

- Two parallel planes: given  $\mathbb{P}_1$  and  $\mathbb{P}_2$  parallel, let  $P \in \mathbb{P}_1$  be any point. Then  $d(\mathbb{P}_1, \mathbb{P}_2) = d(P, \mathbb{P}_2)$  (for which we have a formula from Lecture 4).
- Two skew lines: view  $\ell_1$  and  $\ell_2$  (with directions  $\vec{v}_1$  and  $\vec{v}_2$ ) as lying in parallel planes  $\mathbb{P}_1$  and  $\mathbb{P}_2$ , with the same normal vector  $\vec{n} = \vec{v}_1 \times \vec{v}_2$ . Then  $d(\ell_1, \ell_2) = d(\mathbb{P}_1, \mathbb{P}_2)$ , which we already know how to find.

**Cylinders.** This is the simplest example of a surface other than a plane. You take a plane  $\mathbb{P}$ , a curve  $C \subset \mathbb{P}$ , and a vector  $\vec{v}$  not in  $\mathbb{P}$ . Then draw all the lines with direction vector  $\vec{v}$ , through every point of  $C$ . This gives a surface  $S$  called a cylinder. In most examples you'll see, the vector  $\vec{v}$  is usually  $\hat{i}$ ,  $\hat{j}$ , or  $\hat{k}$ , and so  $x$ ,  $y$ , or  $z$  just doesn't appear in the equation.

**Quadratics.** Given by equations of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

After translation and rotation (change of coordinates), these may all be brought into the form (I)  $Ax^2 + By^2 + Cz^2 = K$  or (II)  $Ax^2 + By^2 = Mz$ . If  $A$ ,  $B$ ,  $C$ , or  $K$  is zero, we get a cylinder. Otherwise,

- Case (I) leads to:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  (ellipsoid),  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$  (hyperboloid of 1-sheet),  $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$  (hyperboloid of 2 sheets), or  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$  (elliptic cone), depending on the signs of  $A, B, C$ , etc.
- Case (II) leads to:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$  (elliptic paraboloid) or  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$  (hyperbolic paraboloid).

You should be able to use “traces” to sketch the surface, and complete the square to put the simpler quadric equations into one of these standard forms. (We won’t deal with the equations that require a “rotation”.)