MATH 233 LECTURE 8 (§13.2): CALCULUS FOR VECTOR-VALUED FUNCTIONS

- One may regard a vector-velued function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ as describing the position of an object in space (pointing from the origin to the point $(f(t), g(t), h(t))$, i.e. *as tracing out a curve C in $\mathbb{R}^3$*.
- We call its derivative $\vec{r}'(t)$ the *tangent* or *veclocity vector*, $\|\vec{r}'(t)\|$ the *speed*, and $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ the *unit tangent vector* (defined so long as $\vec{r}'(t) \neq \vec{0}$).
- The tangent line to C at $\vec{r}(a)$ is represented (traced out, parametrized) by

$$\vec{\ell}_a(u) = \vec{r}(a) + u\vec{r}'(a) = \langle f(a) + f'(a)u, g(a) + g'(a)u, h(a) + h'(a)u\rangle.$$

  Here a different parameter u is used instead of t because we have set $t = a$.
- The curve C is *smooth* at $\vec{r}(a)$ if and only if $\lim_{t\to a}\vec{T}(t)$ exists. We say C is smooth if it is smooth at every point. A smooth curve C in $\mathbb{R}^3$ can be parametrized by a continuously differentiable vector function $\vec{r}(t)$ whose speed is never zero. Roughly, this means that the curve has no corners or cusps.
- Rules for differentiating: given vector functions $\vec{u}(t)$ and $\vec{v}(t)$, we have Leibniz rules $(\vec{u}\cdot\vec{v})' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$, $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$, and the Chain rule $[\vec{u}(F(t))]' = F'(t)\vec{u}'(F(t))$.
- Integrals: $\int_a^b \vec{r}(t)dt = \langle\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\rangle$. If $\vec{v}(t) = \vec{r}'(t)$, then $\int_a^b \vec{v}(t)dt = \vec{r}(b) - \vec{r}(a) =: \vec{r}(t)|_a^b$ (Fundamental theorem).