MATH 233 MIDTERM EXAM 1 ANSWER KEY

This exam consists of 15 multiple choice (machine-graded) problems, worth 4 points each (for a total of 60 points), and 2 pages of written (hand-graded) problems, worth a total of 40 points. No 3x5 cards or calculators are allowed.

PART I: MULTIPLE CHOICE PROBLEMS (Avg. score ~ 37/60)

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. **Shade in the corresponding boxes below.** Also **print your name at the top of your card.**

(1) Find the area of the triangle with vertices $(0,0,0)$, $(\frac{2}{3},-\frac{2}{3},\frac{1}{3})$, and $(1,2,2)$.

(A) $\frac{1}{2}$
(B) 1
(C) $\frac{3}{2}$
(D) 2
(E) $\frac{5}{2}$
(F) 3
(G) $\frac{7}{2}$
(H) 4
(I) none of the above

$$\frac{1}{2}\left\|\left\langle \frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right\rangle \times \langle 1, 2, 2\rangle\right\| = \frac{1}{2}\|\langle -2,-1,2\rangle\| = \frac{3}{2}$$

(2) Which plane is parallel to the one containing the triangle of problem (1)?

(A) $2(x-1)+(y-2)+2(z-2)=1$
(B) $-2x+y+2z=-1$
(C) $-4x+y+z=3$
(D) $(x-\frac{2}{3})+2(y+\frac{2}{3})+2(z-\frac{1}{3})=0$
(E) $2x+y-2z=5$
(F) none of the above

use $\vec{n} = \langle -2,-1,2\rangle$ (or any multiple thereof)

(3) Find the angle between the planes $x+y+2z=4$ and $x+z=-2$.

(A) $30°$
(B) $45°$
(C) $60°$
(D) $90°$
(E) parallel
(F) $-20°F$
(G) none of the above

$\vec{n}_1 = \langle 1,1,2\rangle$ $\quad \vec{n}_2 = \langle 1,0,1\rangle$

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°$$

(4) What are equations for the line at the intersection of the two planes in problem (3)?

(A) $x=-1-t,\ y=-t,\ z=-1+t$
(B) $x=t,\ y=8-t,\ z=-2-t$
(C) $x=-2+t,\ y=6+t,\ z=-t$
(D) $x=-8-t,\ y=t,\ z=6+t$
(E) all of the above
(F) none of the above

either plug in or use $\vec{n}_1\times\vec{n}_2 = \langle 1,1,-1\rangle$: any multiple of this = direction vector

(5) Determine the distance between the point $(4,-2,3)$ and the plane $4x-4y+2z=2$.

(A) 1
(B) 2
(C) 5
(D) $\frac{10}{3}$
(E) $\frac{14}{3}$
(F) $\frac{16}{3}$
(G) none of the above

$$\frac{|4\cdot4-4(-2)+2\cdot3-2|}{\sqrt{4^2+(-4)^2+2^2}}=\frac{28}{6}=\frac{14}{3}$$

(6) Which is an equation for the plane parallel to the plane of problem (5) and through $(4,-2,3)$?

(A) $4x-4y+2z=-15$
(B) $-2x+2y-z=-15$
(C) $4x-4y+2z=14$
(D) $4x-2y+3z=30$
(E) all of the above
(F) none of the above

need $\vec{n}$ = multiple of $\langle 4,-4,2\rangle$

*(7) Find the equation (in x,y,z) of the cone on which $\vec{r}(t)=\langle t\sin t,\, t\cos t,\, \sqrt{8}t\rangle$ lies. Which point lies on this cone?

(A) $(3,-4,-10\sqrt{2})$
(B) $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\sqrt{8})$
(C) $(\sqrt{2},0,-4)$
(D) $(-\sqrt{2},4,12)$
(E) all of the above
(F) none of the above

$x^2+y^2=t^2=\left(z/\sqrt{8}\right)^2=z^2/8$

all 4 points satisfy this

(8) Identify $r=\frac{6}{2+\sin\theta}$ as a conic.

(A) hyperbola centered at $(0,2)$
(B) parabola with vertex $(0,-2)$
(C) ellipse with center $(0,-2)$
(D) parabola with vertex $(2,0)$
(E) circle with center $(0,-2)$
(F) ellipse with center $(2,0)$
(G) asymptotic hypotenuse
(H) polar bear
(I) none of the above

$2r+\underbrace{r\sin\theta}_{y}=6$

$2r=6-y$

square

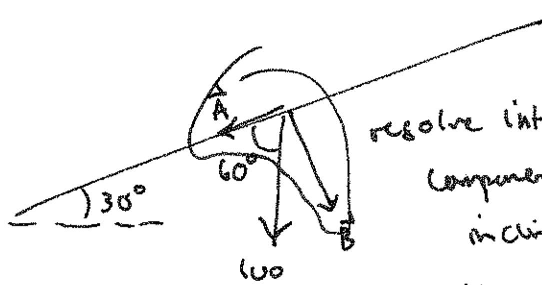
$4x^2+4y^2=36-12y+y^2$

$4x^2+3y^2+12y-36=0$

$4x^2+3(y+2)^2=48$

(9) A 100 N weight sits on a 30° incline. How much force, applied parallel to the surface of the incline, will prevent it sliding down?

(A) 0 N
(B) 50 N
(C) $50\sqrt{3}$ N
(D) 100 N
(E) $100\sqrt{3}$ N
(F) 200 N
(G) none of the above



resolve into components. The incline counteracts $\vec{B}$. You need to counteract $\vec{A}$ which has $\|\vec{A}\|=100\cos 60°=50$.

(10) Find the center and radius of $x^2 + y^2 + z^2 - 4x + 10y - 8z = 36$.

(A) center $(4, -10, 8)$, radius 6
(B) center $(2, -5, 4)$, radius 9
(C) center $(-4, 10, -8)$, radius 3
(D) center $(-2, 5, -4)$, radius 6
(E) center $(4, -10, 8)$, radius 9
(F) center $(0, 0, 0)$, radius 3
(G) center $(2, -5, 4)$, radius 6
(H) none of the above

$(x-2)^2 + (y+5)^2 + (z-4)^2 = 9^2$

(11) Consider a box with dimensions $1ft \times 1ft \times 4ft$. Let θ be the angle between the diagonal of the box and the diagonal of the $1ft \times 1ft$ side. What is $\cos\theta$?

(A) 0
(B) $\frac{1}{6}$
(C) $\frac{\sqrt{3}}{2}$
(D) $\frac{1}{4}$
(E) $\frac{1}{3}$
(F) $\frac{1}{2}$
(G) $\frac{2}{\sqrt{3}}$
(H) $-\frac{1}{6}$
(I) none of the above

$$\cos\theta = \frac{\langle 1,1,0\rangle \cdot \langle 1,1,4\rangle}{\sqrt{1^2+1^2}\,\sqrt{1^2+1^2+4^2}} = \frac{2}{\sqrt{2}\sqrt{18}} = \frac{2}{6} = \frac{1}{3}$$

(12) Calculate the vector projection of $\vec{b} = \langle -3, 2\rangle$ onto $\vec{a} = \langle 3, 4\rangle$.

(A) $\langle -\frac{3}{25}, -\frac{4}{25}\rangle$
(B) $\langle \frac{3}{5}, \frac{4}{5}\rangle$
(C) $\langle -3, -4\rangle$
(D) $-\frac{1}{5}$
(E) $\langle \frac{3}{5}, \frac{4}{5}\rangle$
(F) $\frac{1}{5}$
(G) $\langle 3, 4\rangle$
(H) none of the above

scalar proj. $= \langle -3, 2\rangle \cdot \frac{\vec{a}}{\|\vec{a}\|} = \langle -3, 2\rangle \cdot \frac{\langle 3,4\rangle}{5} = -\frac{1}{5}$

vector proj. $= \left(-\frac{1}{5}\right) \cdot \frac{\vec{a}}{\|\vec{a}\|} = -\frac{1}{5}\left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \left\langle -\frac{3}{25}, -\frac{4}{25}\right\rangle$

* (13) Which are equations of the line through $(0, 3, 8)$ and $(-1, 4, 6)$?

(A) $x = -t$, $y = 3 + 4t$, $z = 8 + 6t$
(B) $x = -2 + 3t$, $y = 5 - 3t$, $z = 4 + 6t$
(C) $x - 1 = 4 - y = \frac{z-6}{2}$
(D) $x = y - 3 = \frac{z-8}{2}$
(E) $x = -1 + 2t$, $y = 4 - 2t$, $z = 6 + 2t$
(F) all of the above
(G) none of the above

$\vec{v}$ must be a multiple of
$\langle 0-(-1), 3-4, 8-6\rangle = \langle 1, -1, 2\rangle$.
Only (B) & (C) pass this test.
(C) doesn't contain the points; (B) does.

(14) Which expression is meaningless?

(A) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
(B) $(\vec{a} \times \vec{b}) \times \vec{c}$
(C) $\cancel{(\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})}\vec{b}$
(D) $\vec{a} \times (\vec{b} \cdot \vec{c})$
(E) $\vec{a} \cdot (\vec{b} \times \vec{c})$
(F) none of the above
(G) all of the above
(H) all mathematical expressions

$\vec{b} \cdot \vec{c}$ is a scaler

(15) Find the distance between the lines $\frac{x-3}{2} = \frac{y+2}{-2} = z - 1$ and $x + 4 = \frac{y+5}{2} = \frac{z}{2}$.

(A) zero: they intersect
(B) 1
(C) 2
(D) 3
~~(E) 4~~
(F) 5
~~(G) 6~~
(H) 7
(I) none of the above

$\vec{v}_1 = \langle 2, -2, 1 \rangle \qquad P_1 = (3, -2, 1)$

$\vec{v}_2 = \langle 1, 2, 2 \rangle \qquad P_2 = (-4, -5, 0)$

$\overrightarrow{P_1P_2} = \langle -7, -3, -1 \rangle$

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle -6, -3, 6 \rangle \rightarrow$ replace with $\langle -2, -1, 2 \rangle$

$$\text{dist} = \frac{|\vec{n} \cdot \overrightarrow{P_1P_2}|}{\|\vec{n}\|} = \frac{|14 + 3 - 2|}{\sqrt{2^2+1^2+2^2}} = \frac{15}{3} = 5.$$

Student Name: ANSWER KEY

MATH 233 MIDTERM EXAM 1

PART II: HAND-GRADED PROBLEMS

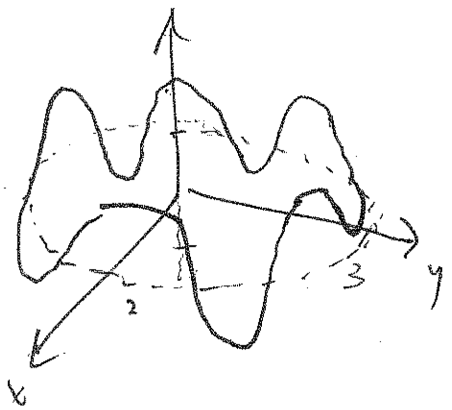
This part has two pages. Show all the work you want graded for each problem in the space provided. Please **print your name at the top of each page.**

(1) You think you're about to take a calculus test in Colorado, but then you notice you're on a roller-coaster ride in the intergalactic space station. Suddenly you decide you need to go to the intergalactic lavatory (located for your convenience at $(-4, 0, 7)$). Since there's no gravity, at the instant you release your seat-belt you will fly off in the tangent direction. Let's figure out how to time this.

(a) [5 points] Write an abstract formula for the tangent line to $\vec{r}(t)$ [= the roller-coaster] at $t = a$. Let me suggest using some variable besides t (say u) to parametrize the tangent line, as in $\vec{\ell}_a(u)$.

$$\vec{\ell}_a(u) = \vec{r}(a) + \vec{r}\,'(a)\,u$$

(b) [8 points] Let's say $\vec{r}(t) = \langle 2\cos t, 3\sin t, \cos 5t\rangle$. Draw a very rough sketch of the curve this traces out, and find $\vec{\ell}_a(u)$ by plugging in your answer to part (a).



$$\vec{\ell}_a(u) = \langle 2\cos a, 3\sin a, \cos 5a\rangle + u\langle -2\sin a, 3\cos a, -5\sin 5a\rangle$$
$$= \boxed{\langle 2\cos a - (2\sin a)u,\ 3\sin a + (3\cos a)u,\ \cos 5a - (5\sin 5a)u\rangle}$$

(c) [7 points] Solve for a value of a which makes the tangent line $\vec{\ell}_a(u)$ pass through the lavatory. (This is when you should self-eject.) [Note: this part is a little tricky. If you're running short on time, try the next page first.]

$$\begin{cases} 2\cos a - (2\sin a)u = -4 & (1) \\ 3\sin a + (3\cos a)u = 0 & (2) \\ \cos 5a - (5\sin 5a)u = 7 & (3) \end{cases}$$

(2) $\Rightarrow$ $u = -\tan a$

w/ (1) $\Rightarrow$ $2\cos a + 2\sin a \cdot \frac{\sin a}{\cos a} = -4 \Rightarrow \underbrace{2\cos^2 a + 2\sin^2 a}_{2} = -4\cos a$

$\Rightarrow -\frac{1}{2} = \cos a \Rightarrow \boxed{a = \frac{2\pi}{3}}$

(Now (3) $\Leftrightarrow$ $\cos\frac{10\pi}{3} - \left(5\sin\frac{10\pi}{3}\right)\left(-\tan\frac{2\pi}{3}\right) = -\frac{1}{2} - 5\left(-\frac{\sqrt{3}}{2}\right)(-\sqrt{3}) = 7$, which works.)

Student Name: ANSWER KEY

(2) (a) [6 points] Find the equation of the surface consisting of all points equidistant from the xz-plane and the point $(0, 2, 0)$.

$P=(x,y,z)$ $\quad d(P,(0,2,0)) = d(P, xz\text{-plane})$

$$\sqrt{x^2+(y-2)^2+z^2} = y$$

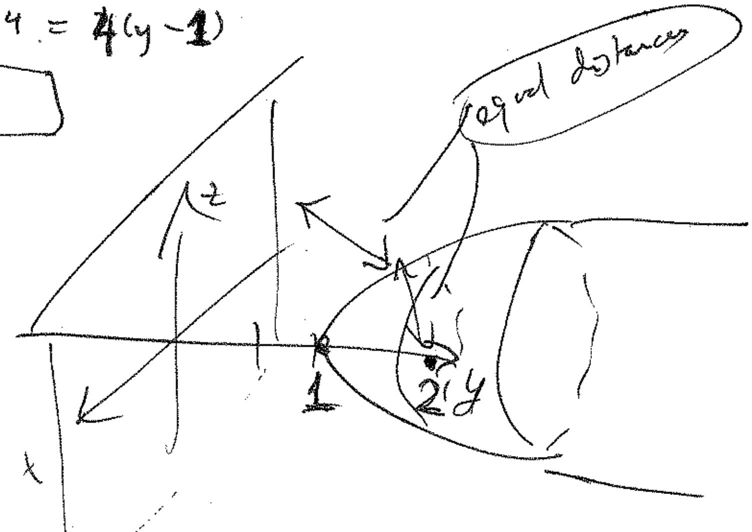
$$x^2+\cancel{y^2}-4y+4+z^2=\cancel{y^2}$$

$$x^2+z^2 = 4y-4 = 4(y-1)$$

$$\boxed{y-1 = \frac{x^2+z^2}{4}}$$

(b) [6 points] Identify and sketch the surface.

elliptic paraboloid with vertex at $(0,1,0)$



(3) [8 points] Let $\vec{r}(t) = \langle 1+\frac{t^4}{4}, \frac{\sqrt{2}t^3}{3}, \frac{t^2}{2}\rangle$. Find the unit tangent vector $\vec{T}(t)$ and use this to decide whether the space curve traced out by $\vec{r}(t)$ is everywhere smooth. [Hint/Warning: you really do need $\vec{T}(t)$.]

Not smooth.

$$\vec{r}\,'(t) = \langle t^3, \sqrt{2}t^2, t\rangle$$

Important

$$\|\vec{r}\,'(t)\| = \sqrt{t^6+2t^4+t^2} = |t|(t^2+1)$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{t\langle t^2, \sqrt{2}t, 1\rangle}{|t|\,(t^2+1)} = \frac{t}{|t|}\underbrace{\left\langle \frac{t^2}{t^2+1}, \frac{\sqrt{2}t}{t^2+1}, \frac{1}{t^2+1}\right\rangle}_{\text{limits to } \langle 0,0,1\rangle \text{ as } t\to 0}$$

$$\lim_{t\to 0^+}\vec{T}(t) = \langle 0,0,1\rangle$$

$$\neq \langle 0,0,-1\rangle = \lim_{t\to 0^-}\vec{T}(t)$$

(So curve is singular at $\vec{r}(0) = \langle 1,0,0\rangle$)