MATH 233 FINAL EXAM ANSWER KEY

This exam consists of 12 multiple choice (machine-graded) problems, worth 5 points each (for a total of 60 points), and 2 pages of written (hand-graded) problems, worth a total of 40 points. No 3x5 cards or calculators are allowed.

## PART I: MULTIPLE CHOICE PROBLEMS

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. **Shade in the corresponding boxes below.** Also **print your name at the top of your card.**

(1) Find the volume of the parallelepiped with edges $\langle 3,2,1\rangle$, $\langle 1,1,2\rangle$ and $\langle 1,3,3\rangle$.
(A) 11
(B) 7
(C) 3
(D) -1
(E) -5
(F) -9

$$\underbrace{\begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{vmatrix}}_{\text{scalar triple product}} = 3\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3(-3) - 2(1) + 1(2)$$

$$= -9$$

(2) Consider the curve traced out by $\vec{r}(t) = \langle 8\cos t,\, 6t,\, 8\sin t\rangle$ for $-5 \le t \le 5$. Find the total arclength.
(A) 10
(B) 20
(C) 50
(D) 60
(E) 100
(F) 200

$$\vec{r}'(t) = \langle -8\sin t, 6, 8\cos t\rangle$$

$$\|\vec{r}'(t)\| = \sqrt{8^2\sin^2 t + 6^2 + 8^2\cos^2 t} = \sqrt{8^2+6^2} = 10.$$

$$\ell = \int_{-5}^{5} 10\,dt = 100.$$

(3) For $\vec{r}(t)$ as in (2), compute the radius of the osculating circle (at any point).
(A) $\frac{25}{2}$
(B) 8
(C) $\frac{5}{4}$
(D) $\frac{4}{5}$
(E) $\frac{1}{8}$
(F) $\frac{2}{25}$

$$\hat{T} = \frac{\vec{r}'}{\|\vec{r}'\|} = \left\langle -\frac{4}{5}\sin t, \frac{3}{5}, \frac{4}{5}\cos t\right\rangle$$

$$\kappa = \frac{\|\hat{T}'\|}{\|\vec{r}'\|} = \frac{1}{10}\left\|\left\langle -\frac{4}{5}\cos t, 0, -\frac{4}{5}\sin t\right\rangle\right\| = \frac{2}{25}$$

$$R = \frac{1}{\kappa} = \frac{25}{2}$$

(4) On a distant planet, gravity is $2\,m/s^2$. Determine the speed (in $m/s$) at which a projectile must be thrown at an angle of $30°$ above the horizontal, from a $10\,m$ high tower, to hit an object $90\sqrt{3}\,m$ away. ✓ (on the ground)

(A) 1
(B) 3
(C) 9
(D) 18
(E) 27
(F) 81

$\vec{a} = \langle 0,-2\rangle$ $\int$ — speed at $t=0$

$\vec{v} = \vec{v}_0 + \langle 0,-2t\rangle = s_0\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle + \langle 0,-2t\rangle$ ($\cos 30°$, $\sin 30°$)

$\vec{r} = \vec{r}_0 + t s_0 \langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle + \langle 0, -t^2\rangle = \langle \frac{\sqrt{3}}{2}s_0 t, 10 + \frac{s_0}{2}t - t^2\rangle$ ($\vec{r}_0 = \langle 0,10\rangle$)

Solve $\begin{cases} \frac{\sqrt{3}}{2} s_0 t_0 = 90\sqrt{3} \rightsquigarrow t_0 = 180/s_0 \\ 10 + \frac{s_0}{2}t_0 - t_0^2 = 0 \rightsquigarrow 0 = 10 + \frac{\cancel{s_0}}{\cancel{2}}\frac{\cancel{180}^{90}}{\cancel{s_0}} - \frac{180^2}{s_0^2}\end{cases}$

$\Rightarrow 100 = \frac{180^2}{s_0^2} \Rightarrow s_0^2 = \frac{180^2}{10^2} = 18^2$

$\Rightarrow s_0 = 18$

(5) Solid gold is pouring out of a slot machine into a conical pile, in such a way that at a certain instant, the height $h$ is $9\,in$ and increasing at $3\,in/min$, and the radius $r$ is $4\,in$ and increasing at $2\,in/min$. How fast (in $in^3/min$) is the volume increasing at that instant? [Hint: $V = \frac{\pi}{3}r^2h$ for a cone.]

(A) $16\pi$
(B) $32\pi$
(C) $48\pi$
(D) $64\pi$
(E) $80\pi$
(F) $96\pi$

$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2\pi}{3}rh\frac{dr}{dt} + \frac{\pi}{3}r^2\frac{dh}{dt}$

$= \frac{2\pi}{\cancel{3}}\cdot 4\cdot \cancel{9}^{3}\cdot 2 + \frac{\pi}{\cancel{3}}\cdot 4^2\cdot \cancel{3} = 64\pi.$

(6) Set $f(x,y) = \frac{2}{9}y^{\frac{3}{2}} + \frac{1}{6}xy$. Find the angle between the $xy$-plane and the tangent plane to $z = f(x,y)$ at $(0,2,f(0,2))$.

(A) 0
(B) $\frac{\pi}{6}$
(C) $\frac{\pi}{5}$
(D) $\frac{\pi}{4}$
(E) $\frac{\pi}{3}$
(F) $\frac{\pi}{2}$

$f_x = y/6, \quad f_y = \frac{1}{3}\sqrt{y} + x/6$

$\vec{n} = \langle -f_x(0,2), -f_y(0,2), 1\rangle = \langle -1/3, -\sqrt{2}/3, 1\rangle$

$\cos\theta = \frac{\langle 0,0,1\rangle\cdot\vec{n}}{\cancel{\|\langle 0,0,1\rangle\|}\cdot\|\vec{n}\|} = \frac{1}{\|\vec{n}\|} = \frac{1}{\sqrt{\frac{1}{9}+\frac{2}{9}+1}}$

$= \frac{1}{\sqrt{12/9}} = \sqrt{\frac{3}{4}} = \sqrt{3}/2$

$\Rightarrow \theta = \pi/6.$

(7) Compute the directional derivative of $f(x, y, z) = xy + z^2$ at $(1, 1, 1)$ in the direction *toward* $(5, -3, 3)$ from there.

(A) 0
(B) $\frac{1}{3}$
(C) $\frac{2}{3}$
(D) 1
(E) $\frac{4}{3}$
(F) 2

$$\vec{u} = \langle 5-1, -3-1, 3-1 \rangle = \langle 4, -4, 2 \rangle$$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{u}}{6} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$(D_{\hat{u}}f)(1,1,1) = \hat{u} \cdot \underbrace{\vec{\nabla} f}_{\langle y, x, 2z \rangle}(1,1,1) = \hat{u} \cdot \langle 1, 1, 2 \rangle$$

$$= \frac{2}{3} \cdot 1 - \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2 = \frac{2}{3}.$$

(8) If $T(x, y, z) = 2x^2 + y^2 + z^2$ is the temperature function (in °$C$) on the disk $\underbrace{x^2 + (y-2)^2 + z^2}_{g} \leq 9$, what are the hottest and coldest temperatures on the disk?

(A) 24; 0
(B) 25; 0
(C) 26; 0
(D) 24; 1
(E) 25; 1
(F) 26; 1

interior: $\vec{\nabla} T = 0 \Rightarrow (x, y, z) = (0, 0, 0)$

boundary: $\vec{\nabla} T = \lambda \vec{\nabla} g \Rightarrow 2x = \lambda x, \; y = \lambda(y-2), \; z = \lambda z$

If $\lambda = 1$, then $y = y - 2$ ✻. So $z = 0$, and $x^2 + (y-2)^2 = 9$. Next, $x = 0$ or $\lambda = 2$.

If $x = 0$, then $y = 5$ or $-1$. If $\lambda = 2$, then $y = 2y - 4 \Rightarrow y = 4 \Rightarrow x = \pm\sqrt{5}$.

$T(0,0,0) = 0$ min

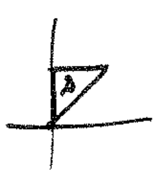
$T(\pm\sqrt{5}, 4, 0) = 26$ max

$T(0, 5, 0) = 25$

$T(0, -1, 0) = 1$.

(9) Find $\iint_{\mathcal{D}} \frac{2}{1+x^2} dA$, where $\mathcal{D}$ is the triangular region with vertices at $(0, 0)$, $(1, 1)$ and $(0, 1)$.

(A) $\frac{\pi}{2}$
(B) $\pi$
(C) $\ln 2$
(D) $2 \ln 2$
(E) $\frac{\pi}{2} - 2 \ln 2$
(F) $\frac{\pi}{2} - \ln 2$



$$\int_0^1 \int_x^1 \frac{2}{1+x^2} \, dy \, dx =$$

$$\int_0^1 \left[ \frac{2y}{1+x^2} \right]_{y=x}^{1} dx = \int_0^1 \frac{2}{1+x^2} dx - \int_0^1 \frac{2x}{1+x^2} dx$$

$$= \left[ 2\arctan x \right]_0^1 - \left[ \ln(1+x^2) \right]_0^1$$

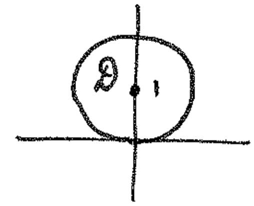
$$= \underbrace{2\arctan 1}_{\pi/4} - \cancel{2\arctan 0} - \ln 2 + \cancel{\ln 1}$$

$$= \pi/2 - \ln 2.$$

(10) Consider the disk of radius 1 with center $(0,1)$ and mass density function $\rho(x,y)=\sqrt{x^2+y^2}$. Compute the total mass.

(A) $\frac{32}{9}$
(B) $\frac{16}{3}$
(C) $\frac{8}{3}$
(D) $\frac{4\pi}{3}$
(E) $\pi$
(F) $\frac{2\pi}{3}$

$$\iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^{2\sin\theta} r\; r\,dr\,d\theta$$




$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{2\sin\theta} d\theta$$

$$= \frac{8}{3}\int_0^{\pi}\sin^3\theta\,d\theta$$

$\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

$$= \frac{8}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \frac{8}{3}\left(2-\frac{2}{3}\right) = \frac{8}{3}\cdot\frac{4}{3} = 32/9.$$

$x^2+(y-1)^2=1$

$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$

$\theta\in[0,\pi]$ traces boundary of $D$

(11) Say we are integrating in $x$ and $y$ and we want to integrate in $u=\frac{x^2}{y}$ and $v=\frac{x^5}{y^2}$. We must replace $dx\,dy$ by what function times $du\,dv$?

(A) $\frac{5v^2}{u^8}$
(B) $\frac{v^2}{u^8}$
(C) $\frac{5x^6}{y^4}$
(D) $\frac{x^6}{y^4}$
(E) $\frac{5u^8}{v^2}$
(F) $\frac{u^8}{v^2}$

$$\frac{v}{u^2} = \frac{x^5/y^2}{x^4/y^2} = x$$

$$\frac{v^2}{u^5} = \frac{x^{10}/y^4}{x^{10}/y^5} = y$$

$$\begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -2v/u^3 & -5v^2/u^6 \\ 1/u^2 & 2v/u^5 \end{vmatrix}$$

$$= -\frac{4v^2}{u^8} + \frac{5v^2}{u^8} = \frac{v^2}{u^8}$$

(12) Let $\vec{F} = \overbrace{(2x+y)}^{P}\hat{\imath} + \overbrace{(x-2y)}^{Q}\hat{\jmath}$. Compute $\int_C \vec{F}\cdot d\vec{r}$, where $C$ is any oriented curve starting at $A=(1,2)$ and ending at $B=(3,0)$.

(A) $-10$
(B) $-5$
(C) 0
(D) 5
(E) 10
(F) the integral is not independent of path

$Q_x = P_y \;(=1)$

$\Rightarrow \vec{F}$ conservative.

Indeed, $\vec{F} = \vec{\nabla}(\underbrace{x^2+xy-y^2}_{=:f})$.

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(B)-f(A) = \underbrace{(3^2+0-0)}_{9} - \underbrace{(1^2+1\cdot 2-2^2)}_{-1}$$

$$= 10.$$

MATH 233 FINAL EXAM

PART II: HAND-GRADED PROBLEMS ANSWER KEY

This part has two pages. Show all the work you want graded for each problem in the space provided. Please **print your name at the top of each page.**

(1) [12 points] Find the work done by the force field $\vec{F}(x, y, z) = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ in moving a particle along the oriented curve $\mathcal{C}$ traced out by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $t \in [0, 1]$.

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C y\,\overbrace{dx}^{x'(t)\,dt} + z\,\overbrace{dy}^{y'(t)\,dt} + x\,\overbrace{dz}^{z'(t)\,dt}$$

$$= \int_0^1 \left(t^2 \cdot 1 + t^3 \cdot 2t + t \cdot 3t^2\right) dt$$

$$= \int_0^1 \left(t^2 + 3t^3 + 2t^4\right) dt$$

$$= \left[\frac{t^3}{3} + \frac{3t^4}{4} + \frac{2t^5}{5}\right]_0^1$$

$$= \frac{1}{3} + \frac{3}{4} + \frac{2}{5}$$

$$= \frac{89}{60}$$

(2) [8 points] Are the integrals $\oint_C \vec{F} \cdot d\vec{r}$ of $\vec{F}(x, y, z) = (\overbrace{2xyz + z^2}^{P})\hat{\imath} + (\overbrace{x^2 z + z^3}^{Q})\hat{\jmath} + (\overbrace{x^2 y + 3yz^2}^{R})\hat{k}$ around any closed path equal to zero? Why or why not?

While $Q_x - P_y$ and $R_y - Q_z$ are $0$,

$R_x - P_z = (2xy) - (2xy + 2z) = -2z \neq 0$

$\Rightarrow$ $\text{curl}\,\vec{F} \neq \vec{0}$

$\Rightarrow$ $\vec{F}$ not conservative

$\Rightarrow$ not all $\oint_C \vec{F} \cdot d\vec{r}$ are $0$.

Answer: NO.

(3) [10 points] Use Gauss's theorem in the plane to compute the flux of $\vec{F}(x,y) = (e^{-y^2}+2x)\hat{\imath}+(e^{-2x^2}+y)\hat{\jmath}$ across the (counterclockwise oriented) boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

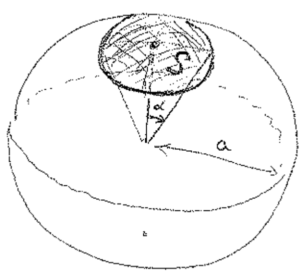

$$\oint_{C} \vec{F}\cdot\hat{n}\,ds = \iint_{D} \operatorname{div}\vec{F}\,dA$$

$$= \iint_{D}\Big(\underbrace{\frac{\partial}{\partial x}(e^{-y^2}+2x)}_{2}+\underbrace{\frac{\partial}{\partial y}(e^{-2x^2}+y)}_{1}\Big)\,dA$$

$$= 3\iint_{D} dA$$

$$= 3\,\text{Area}(D) = 3/2$$

(4) [10 points] Determine a formula for the surface area of the "polar cap" on a sphere of radius $a$ determined by the spherical angle $\alpha$. (For full credit you must compute the integral; of course, the final expression should involve $a$ and $\alpha$.)




$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle,\quad \phi\in[0,\alpha],\ \theta\in[0,2\pi]$$

$$\vec{r}_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0\rangle$$

$$\vec{r}_\phi\times\vec{r}_\theta = \langle -a^2\sin^2\phi\cos\theta,\ -a^2\sin^2\phi\sin\theta,\ a^2\cos\phi\sin\phi\rangle$$

$$\|\vec{r}_\phi\times\vec{r}_\theta\| = \sqrt{a^4\sin^4\phi\,\cancel{(\cos^2\theta+\sin^2\theta)}+\cancel{a^4\sin^2\phi\cos^2\phi}}$$

$$\sin^2\phi(1-\cancel{\cos^2\phi})$$

$$= a^2\sin\phi$$

$$\text{Area}(S) = \int_0^{2\pi}\int_0^{\alpha}\underbrace{\|\vec{r}_\phi\times\vec{r}_\theta\|}_{a^2\sin\phi}\,d\phi\,d\theta = 2\pi a^2\int_0^{\alpha}\sin\phi\,d\phi$$

$$= 2\pi a^2\big[-\cos\phi\big]_0^{\alpha} = 2\pi a^2(1-\cos\alpha).$$