Exam 1 Review

Format: 10 multiple-choice questions (50 pts.)
2 pp. worth of free response questions (50 pts.)

Bring: pencils + ID (no calculators or 3x5 cards)

Remarks: Seating is not assigned
Early exam is 4:20-6:20 in McDonnell 162
regular exam is 6:30-8:30 in Lab Sciences 300

Material: Chapters 1-3, minus all "applied" material like
Leontief model, chemical equations, network flow, etc.,
as well as the block matrix & LU-factorization stuff.
More precisely: sections 1.1-5, 1.7-9, 2.1-3, 3.1-3.

Types of Questions: Over half of the MC questions require
no computation at all, and are of a similar flavor

to the T/F questions in the book. The remaining ones
require very short computations (like easiest undergrad problems).
The free-response questions require you to do computations -
you can bet on some row-reduction & determinant calculation.
- **Linear Systems**: Know how to convert between

  System of Linear Equations  Vector Equation  Matrix Equation

  \[
  \begin{align*}
  a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
  \end{align*}
  \]

  \[
  \begin{pmatrix} a_{11} & \cdots & a_{1n} \end{pmatrix} \begin{pmatrix} x_1 \\
  \vdots \\
  x_n \end{pmatrix} = \begin{pmatrix} b_1 \\
  \vdots \\
  b_m \end{pmatrix}
  \]

  \[
  A \mathbf{x} = \mathbf{b}
  \]

  If you are asked to write "a system of linear equations", this is what is meant, not one of the other two.

  \[A \mathbf{x} = \mathbf{b}\] consistent means a solution \( \mathbf{x} \) exists (equivalently, \( \mathbf{b} \) can be written as a linear combination of the column vectors).

- **Row-reduction, REF, RREF**: Know how to do the

  Replace, Swap, \& Scale operations, as well as the corresponding elementary matrices, and the effects on the determinant (of a square matrix). Be able to use these operations to get to RREF, and thereby identify the pivot \& non-pivot columns; and to solve matrix equations (apply them to \([A \mid \mathbf{b}]\), then parametrize the solution set by the non-pivot variables). You should know how to identify REF \& RREF matrices, and enumerate the possibilities for any dimension \( m \times n \).
Linear Independence & Span: Vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^m$ are linearly independent if the only way of writing $\vec{0}$ as a linear combination of them is $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. The basic test for independence is to reduce \( \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \) to RREF and check that every column is a pivot column. This is not possible if $n > m$.

The span of these vectors is the set span $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of all linear combinations of them; to decide whether $\vec{b}$ lies in this span, just determine whether $(\frac{\vec{b}}{c_1, \ldots, c_n})$ or $\vec{b}$ is consistent. They "span $\mathbb{R}^m$" if this span $= \mathbb{R}^m$, equivalently if every row of RREF$(\vec{v}_1 \ldots \vec{v}_n)$ has a leading 1; this is not possible if $n < m$.

Matrix products: Given $A = [a_{ij}], B = [b_{jk}]$, $AB$ has entries $c_{ik} = \sum_{j=1}^{m} a_{ij}b_{jk}$. But a (sometimes) better way to look at this is in terms of columns: if $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$, then $A\vec{x} = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$; while if $B = \begin{pmatrix} \vec{w}_1 & \cdots & \vec{w}_r \end{pmatrix}$, $AB = \begin{pmatrix} \vec{w}_1 & \cdots & \vec{w}_r \end{pmatrix}$ where each $A\vec{w}_k$ is a linear combination of columns of $A$. So $A\vec{x} = \vec{b}$ says "$\vec{b}$ is in the span of $A$'s columns"; and if the columns of $B$ are linearly independent, so are those of $AB$. 
* Invertibility and Matrix Inverses: Given an n x n matrix A, we say A is invertible if there is an n x n matrix B such that AB = In or (equivalently) BA = In, where In is the n x n "identity matrix" \( \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \). [Notation: the columns of In are called \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \).] This matrix B is called the inverse of A and written \( A^{-1} \). A is invertible \( \iff \) \( \text{ref}(A) = In \), in which case taking ref of \( [A | In] \) gives \( [In | A^{-1}] \). Keeping track of the elementary matrices \( E_1, \ldots, E_s \) corresponding to the EROs performed in this process allows you to write \( A = E_s^{-1} \cdots E_1^{-1} \).

Finally, \( (AB)^{-1} = B^{-1}A^{-1} \) if both \( A \) and \( B \) are invertible.

* Matrix Transpose: \( A^T \) has \( i,j \)-th entry the \( j,i \)-th entry of \( A \).

We have \( (AB)^T = B^T A^T \), and \( (A^T)^{-1} = (A^{-1})^T \) (if \( A \) invertible)

* Linear Transformations are functions \( T: \mathbb{R}^n \to \mathbb{R}^m \) that "preserve linear combinations": \( T(ax + by) = cT(x) + bT(y) \). Know what "onto" and "1-to-1" mean. The standard matrix \( A \) of \( T \) has its columns given by \( T(\vec{e}_1), \ldots, T(\vec{e}_n) \). \( T \) is 1-1 \( \iff \) columns of \( A \) are independent

\( T \) is onto \( \iff \) columns of \( A \) span \( \mathbb{R}^m \)
If $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is another L.T., with matrix $B$, then $T \circ S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has matrix $AB$. (Note that $B$ is applied first.) You should be able to reproduce the examples in pp. 73-76 of the book, e.g. rotations, projections, and restrictions in $\mathbb{R}^2$ (and compositions of these).

- **Determinants**: Know the properties — linear in each row and column, changes sign when 2 rows or columns are swapped. Completely unaffected by "replace" operations. For a product of matrices, $\det(AB) = \det(A) \cdot \det(B)$. Be able to compute determinants by Laplace expansion and by using EROS. $A$ is invertible $\iff$ $\det(A) \neq 0$, and so the nontriviality of the determinant also furnishes a test of linear independence of $A$'s columns. If $A$ is the matrix of $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, then $\det(A)$ is the "dilation factor" of $T$, i.e. the ratio $\frac{\text{vol}(T(\Omega))}{\text{vol}(\Omega)}$ for any $\Omega \subset \mathbb{R}^n$; if $P$ is the parallelepiped "spanned" by $A$'s columns, then $\text{vol}(P) = \det(A)$. (On the other hand, Cramer's rule / adjugate matrices are not on the exam.)