Solutions to Math 309 Midterm Exam 1

This exam consists of 10 multiple choice (machine-graded) problems, worth 5 points each (for a total of 50 points), and 2 pages of written (hand-graded) problems, worth a total of 50 points. No 3x5 cards or calculators are allowed.

Part I: Multiple choice problems

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your ID number on the six blank lines on the top of your answer card, using one blank for each digit. Shade in the corresponding boxes below. Also print your name at the top of your card.

(1) For problems (1)-(2), let $T$ be a linear transformation, with $5 \times 3$ standard matrix $A$. Is the image of $T$ the span of $A$’s columns or rows, and in what space?
   - (A) columns; in $\mathbb{R}^3$
   - (B) rows; in $\mathbb{R}^3$
   - (C) columns; in $\mathbb{R}^5$
   - (D) rows; in $\mathbb{R}^5$

   $T : \mathbb{R}^3 \rightarrow \mathbb{R}^5$

(2) If $T$ is 1-to-1, how many “leading ones” does $\text{rref}(A)$ contain?
   - (A) 0
   - (B) 1
   - (C) 2
   - (D) 3
   - (E) 4
   - (F) 5

   $\text{rref}(A) = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   0 & 0 & 0
   \end{pmatrix}$

(3) For problems (3)-(5), suppose you have two matrices: $A$ is $m \times n$, $B$ is $n \times m$, where $n \neq m$; and $AB = I_m$. First, what is the relationship between $n$ and $m$?
   - (A) $m > n$
   - (B) $m < n$
   - (C) either is possible

   Columns of $A$ must span $\mathbb{R}^m$

(4) Are the columns of $A$ independent?
   - (A) yes
   - (B) no
   - (C) maybe (either is possible)

   Since $n > m$, there will be non-pivot columns.

(5) What about the columns of $B$? Are they independent?
   - (A) yes
   - (B) no
   - (C) maybe

   If they were dependent, $A$ times them would be dependent. But this contradicts independence of $\tilde{e}_1, \ldots, \tilde{e}_m$. 
(6) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which first rotates by the angle $\theta$ counterclockwise (about the origin) then reflects about the line $x = y$. What is the lower right-hand corner entry $a_{22}$ of its matrix $A$?

(A) $\cos(\theta)$
(B) $\sin(\theta)$
(C) $-\cos(\theta)$
(D) $-\sin(\theta)$
(E) 0

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
= 
\begin{pmatrix}
\sin(\theta) & \cos(\theta) \\
-\cos(\theta) & -\sin(\theta)
\end{pmatrix}
\]

(7) Consider an $n \times n$ invertible matrix $A$, such that $A$ and $A^{-1}$ both have integer entries. What are the possible values of $\det(A)$?

(A) 1
(B) 1, -1
(C) 0, 1, -1
(D) any integer
(E) any nonzero integer

A has integer entries $\Rightarrow \det(A)$ is an integer $n$

\[A^{-1} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{n}\]

so $n = \pm 1$

(8) Which of the following is incorrect in general if $A$ and $B$ are $n \times n$ invertible matrices?

(A) $(AB)^{-1} = B^{-1}A^{-1}$
(B) $(A + B)^T = A^T + B^T$
(C) $A(B + B^{-1}) = AB + AB^{-1}$
(D) $(A + B)^{-1} = A^{-1} + B^{-1}$
(E) $(AB)^T = B^T A^T$
(9) For problems (9)-(10), I am thinking of some numbers $a, b, c, d, e, f$ such that

$$\text{det} \begin{pmatrix} a & 1 & d \\ b & 1 & e \\ c & 1 & f \end{pmatrix} = 7 \quad \text{and} \quad \text{det} \begin{pmatrix} a & 1 & d \\ b & 2 & e \\ c & 3 & f \end{pmatrix} = 11.$$

Find \( \text{det} \begin{pmatrix} d & 3 & a \\ e & 3 & b \\ f & 3 & c \end{pmatrix} \).

(A) 0  
(B) 7  
(C) -7  
(D) 21  
(E) -21

(10) Find \( \text{det} \begin{pmatrix} a & 3 & d \\ b & 5 & e \\ c & 7 & f \end{pmatrix} \).

(A) 7  
(B) 18  
(C) 21  
(D) 29  
(E) 40
Part II: Hand-graded problems

This part has two pages. Show all the work you want graded for each problem in the space provided. Please print your name at the top of each page.

(1) We want to determine whether the span of \( \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix} \) contains \( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \).

(a) [8 points] What system of linear equations has to be consistent for this to be so?

\[
\begin{align*}
\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 &= 0 \\
2\mathbf{x}_1 + \mathbf{x}_2 + 9\mathbf{x}_3 + 2\mathbf{x}_4 &= 0 \\
3\mathbf{x}_1 + \mathbf{x}_2 + 7\mathbf{x}_3 + \mathbf{x}_4 &= 0 \\
3\mathbf{x}_1 + 2\mathbf{x}_2 - \mathbf{x}_4 &= 0
\end{align*}
\]

(b) [12 points] Now solve the problem. Show (and label) all your work.

\[
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 9 & 2 \\ 3 & 1 & 7 & 1 \\ 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & -1 & 3 & 3 \\ 3 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & -10 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

impossible (c = -3)
(2) (a) [12 points] Compute the inverse of \( A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix} \).

\[
\begin{bmatrix}
 1 & 1 \\
 2 & 3 \\
 3 & 4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
 1 & 1 \\
 0 & 1 \\
 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
 1 & -2 \\
 0 & 1 \\
 0 & -3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix}
\]

(b) [8 points] Write \( A \) as a product of elementary matrices.

\[
E_1 = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{5} \end{pmatrix}
\]

and \( E_4 E_3 E_2 E_1 A = I_3 \). So \( A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \)

\[
= \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}
\]

(3) [10 points] For which (real) values of \( s \) are 
\[
\begin{pmatrix}
 s & s^2 & s^3 \\
 s^2 & s^2 & s^3 \\
 s^3 & s & s^2 \\
\end{pmatrix}
\]
linearly independent? (Explain your work, and state any results you use.)

\[
\det\left(\begin{pmatrix}
 s & s^2 & s^3 \\
 s^2 & s^2 & s^3 \\
 s^3 & s & s^2 \\
\end{pmatrix}\right) = s \left| \begin{array}{cc}
 s^2 & s^3 \\
 s & s^2 \\
\end{array} \right| + s^3 \left| \begin{array}{cc}
 s & s^2 \\
 s^2 & s \\
\end{array} \right| - s^2 \left| \begin{array}{cc}
 s^2 & s^3 \\
 s^2 & s \\
\end{array} \right|
\]

\[
= s(s^5 - s^5) - s^2(s^3 - s^4) + s^3(s^3 - s^6)
\]

\[
= s^6 - s^3 + s^6 - s^6 = -s^6(s^3 - 1)^2.
\]

So they are linearly independent for \( s \neq 0,1 \), since \( \det(A) \neq 0 \) \(\Rightarrow\) \(A\) invertible \(\iff\) Columns linearly independent.