Lecture 13: More on Determinants

In lecture 12, we defined \( \det: \{n \times n\ \text{matrices}\} \to \mathbb{R} \) to be the unique function satisfying:

(i) linearity in each row (with the other rows held fixed)
(ii) antisymmetry in the rows
(iii) \( \det I_n = 1 \).

From these properties, we quickly deduced that:

- If 2 rows are equal, then \( \det A = 0 \).
- If \( A \) is upper/lower triangular, \( \det A = \) product of diagonal entries.
  (in particular, if \( A \) is diagonal, i.e., \( A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \) then \( \det A = a_{11} \cdots a_{nn} \)).
- If \( A_{ij} = 0 \) for \( i \neq j \), then \( \det A = 0 \).
- If \( A \) has a row of 0's, then \( \det A = 0 \).

**Ex 1**

\[
\begin{vmatrix}
1 & 0 & 0 \\
1 & 3 & 1 \\
1 & 0 & 2 \\
1 & -1 & 3
\end{vmatrix}
= 1 \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 3 \\ 1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 \\ 0 \\ -1 \\ 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 \\ -1 \\ 3 \\ -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 \\ 1 \\ 0 \\ -1 \end{vmatrix}
= 3(-3) - 1 \cdot 1 + 1(9+1) + 2(-1-3) = -8.
\]
Determinants and EROs (Elementary Row Operations)

Let \( E \) be an elementary matrix, so that \( \tilde{A} = EA \) is one row operation applied to \( A \).

**Theorem 1:** If \( E \) is a \{replace, swap, scale\} operation, \( \det \tilde{A} = \begin{cases} \det A & \text{if } E \text{ is replace} \\ -\det A & \text{if } E \text{ is swap} \\ \mu \det A & \text{if } E \text{ is scale} \end{cases} \)

**Proof:** Write \( A = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} \).

\[
\det \begin{pmatrix} \cdots & A_i & \cdots \end{pmatrix} = \det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} + \mu \det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} \]

by linearity

\[
\det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} = -\det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} = -\det A \]

by antisymmetry

\[
\det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} = \mu \det \begin{pmatrix} \cdots & \cdots & \cdots \end{pmatrix} = \mu \det A \]

by linearity
Ex 2 / Find $\det \begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{pmatrix}$.

**Ideal:** row-reduce to an upper triangular using only replace & swap.

\[
\begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 6 & 9 & 14 & 29 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{vmatrix} = 9.
\]

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**Determinants and Invertibility**

**Theorem 2:** $\det A \neq 0 \iff A$ is invertible.

**Proof:** ($\iff$): $A$ invertible $\Rightarrow A$ is obtained from $I_n$ by EROs $\Rightarrow \det A = \det I_n \cdot (-1)^{E}$ (Theorem 1)

$\Rightarrow \det A = \det I_n \cdot (-1)^{E}$ (Product of scaling factors) $\Rightarrow 0$.

($\Rightarrow$): $\det A \neq 0 \Rightarrow \det (\text{rref} A) = \det A \neq 0$ (Theorem 1)

$\Rightarrow \text{rref} A$ has no row of 0s

$\Rightarrow \text{rref} A = I_n$

$\Rightarrow A$ invertible.
Ex 3/ we know \( \text{ref} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \neq \begin{pmatrix} I \\ I \\ I \end{pmatrix} \), so
\[
\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0.
\]

Ex 4/ Find all values of \( \alpha \) that make
\[
A = \begin{pmatrix} \alpha & 1 & 1 \\ 0 & \alpha & 1 \\ 4 & 2 & 2 \end{pmatrix}
\]
non-invertible ("Singular").

\[
\det A = \begin{vmatrix} \alpha & 1 & 1 \\ 0 & \alpha & 1 \\ 4 & 2 & 2 \end{vmatrix} = (\alpha-1) \begin{vmatrix} \alpha & 1 \\ 4 & 2 \end{vmatrix} = (\alpha-1)(\alpha^2-4) = (\alpha-1)(\alpha-2)(\alpha+2)
\]

Expand in middle row

\( \Rightarrow \) values are \( \alpha = 1, 2, -2 \).

Determinants and Products

Consider the elementary matrices once more. What are their determinants?

- Replace
  - (appear on lower triangle with 1s on diagonal)
- Switch
  - (get in by swapping in other rows)
- Scale

Looking at the statement of Theorem 1, we see it on
be rephrased as follows:

If $E$ is an elementary matrix, then
\[ \det(EA) = \det(E) \cdot \det(A). \]

[Result of applying $E$ to $A$]

**Theorem 3:** Given any matrices $A$ & $B$, $\det AB = (\det A)(\det B)$.

**Proof:**

**Case 1:** $\det A = 0$. Then $A$ isn't invertible.

If $AB$ had an inverse $C$, then $I_n = (AB)C = A(BC) 
\Rightarrow BC$ is inverse to $A$, a contradiction. So $AB$ isn't invertible, and $\therefore \det AB = 0$.

**Case 2:** $\det A \neq 0$. Then $A$ is invertible, and so may be written as a product of elementary matrices:

$A = E_N \cdots E_1 (I_n)$

$\Rightarrow AB = E_N \cdots E_1 B$.

By repeated application of (6),

$\det A = \det E_N \cdots \det E_1 \det I_n, \det E_N \cdots \det E_1 \det B, \therefore \det AB = \det A \cdot \det B. \square$

**Corollary:** If $A$ is invertible, $\det A^{-1} = \frac{1}{\det A}$.

**Ex 5**

$\det B^{-1} A^9 B = (\det B)^{-1} (\det A)^9 \det B = (\det A)^9$. \[\square\]
(What about \( A + B \)? In fact, we can't say anything about its determinant. It's certainly false that \( \det(A + B) = \det A + \det B \); for example, take \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \) Then \( \det A + \det B = 1+1 = 2 \), but \( A + B \) is the zero matrix.)

Determinants and Column Vectors

I mentioned in Lecture 12 that if you define "\( \det \)" to be the unique alternating multilinear normalized function on \{columns of \( A \)\} then you get the same function as doing it via rows (as we've done). Since transposing takes rows to columns, this statement is equivalent to

**Theorem 4**: \( \det A = \det A^T \).

**Proof**: First, \( \det A \neq 0 \iff \det A^T \neq 0 \), since they are both invertible (with \( (A^T)^{-1} = (A^{-1})^T \)) or both not.

So if they're invertible, we have \( A = E_1 \cdots E_n \) and \( A^T = E_n^T \cdots E_1^T \). By Thm. 3, it suffices to check \( \det E_i = \det E_i^T \). But this is obvious: swap & scale matrices are unchanged by transpose, and replace matrices have determinant 1.
Corollary: (ii) Elementary column operations have the same effect on \( \det \) as EROs.

(iii) Laplace expansion holds for columns: for each \( j \),

\[
\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.
\]

**Ex 6** / If \[
\begin{vmatrix}
 a & b & c \\
 d & e & f \\
 g & h & i
\end{vmatrix} = 2,
\]

\[
\begin{vmatrix}
 a-8g & b-8h & c-8i \\
 d & e & f \\
 g & h & i
\end{vmatrix} = 3
\]

\[
\begin{vmatrix}
 a & b & c \\
 d & e & f \\
 g & h & i
\end{vmatrix} = 3 \cdot 2 = 6.
\]

**Ex 7** / \[
\begin{vmatrix}
 1 & 2 & 2 \\
 2 & 3 & 3 \\
 2 & 3 & 3
\end{vmatrix} = \Delta_n - 1
\]

**Laplace expansion**

\[
\begin{vmatrix}
 1 & 2 & 2 & \cdots \\
 0 & 1 & 2 & \cdots \\
 0 & 0 & 1 & \cdots \\
 0 & 0 & 0 & \cdots
\end{vmatrix} = 1
\]

Determinants and Linear Independence

Recall from lectures 11 and 8 that the following conditions on an \( n \times n \) (square) matrix \( A \) are equivalent:
\( C1 \) A is invertible

\( C2 \) The columns of \( A \) span \( \mathbb{R}^n \)

\( C3 \) The columns of \( A \) are linearly independent

The same goes for rows since columns of \( A \) are rows of \( A^T \).

**Theorem 5:** \( \det A \neq 0 \iff \text{columns are lin. ind. of span } \mathbb{R}^n \)

\( \iff \text{columns are lin. ind. of span } \mathbb{R}^n \)

**Ex 8:** For what values of \( x \) is \( \begin{pmatrix} \alpha \\ \alpha \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ x \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} \) a linearly independent set?

We already solved this: They're independent \( \iff \det \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq 0 \iff x \neq 1, 2, -2. \)