Lecture 13: More on Determinants

In Lecture 12, we defined \( \det: \{ \text{matrices} \} \rightarrow \mathbb{R} \) to be the unique function satisfying

(i) linearity in each row (with the other rows held fixed)
(ii) anti-symmetry in the rows
(iii) \( \det I_n = 1 \).

From these properties, we quickly deduced that

- if 2 rows are equal, then \( \det A = 0 \)
- if \( A \) is upper/lower triangular, \( \det A \) = product of diagonal entries
  (in particular, if \( A \) is diagonal, i.e. \( A = (a_{ij}) \) then \( \det A = a_{11} \cdots a_{nn} \))
- if \( A_{ij} \) = (n-1) by (n-1) matrix obtained by deleting \( i \)th row and \( j \)th column of \( A \),
  \( C_{ij} = (-1)^{i+j} \det A_{ij} \) = \( (i,j) \)th cofactor, then (for any \( i \))
  \( \det A = \sum_{j=1}^{n} a_{ij} C_{ij} \) (Laplace expansion along the \( i \)th row).

Consequently,

- if \( A \) has a row of 0's, then \( \det A = 0 \).

Ex 1/

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 3 & 1 & 0 \\
1 & 0 & 2 & 1 \\
1 & -1 & 3 & 0 \\
\end{bmatrix} =
\begin{bmatrix}
3 & 1 & 0 \\
0 & 2 & 1 \\
-1 & 3 & b \\
-1 & -3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
0 & 2 \\
-1 & 3 \\
-1 & -3 \\
\end{bmatrix}
\]

\[
= 3 \left| \begin{array}{cc}
2 & 1 \\
3 & 0 \\
\end{array} \right| - 1 \left| \begin{array}{cc}
1 & 0 \\
-1 & 0 \\
\end{array} \right| + 1 \left| \begin{array}{cc}
3 & 1 \\
-1 & 3 \\
\end{array} \right| + 2 \left| \begin{array}{cc}
1 & 3 \\
-1 & -3 \\
\end{array} \right|
\]

\[
= 3(-3) - 1 \cdot 1 + 1(9 + 1) + 2(-1 - 3) = -9.
\]
Determinants and ERDs (Elementary Row Operations)

Let $E$ be an elementary matrix, so that $\tilde{A} = E \cdot A$ is one row operation applied to $A$.

**Theorem 1:** If $E$ is a \{replace, swap, scale by $\mu$\} operation, then $\det \tilde{A} = \left\{ \begin{array}{ll} \det A & \text{replace} \\ -\det A & \text{swap} \\ \mu \det A & \text{scale by } \mu \end{array} \right.$

**Proof:** Write $A = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}$.

$$\det \begin{pmatrix} \tilde{A}_1 & \cdots & \tilde{A}_n \end{pmatrix} = \det \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} + \mu \det \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} \overset{\text{since}}{=} \det A$$

by linearity

$$\det \begin{pmatrix} \tilde{A}_1 & \cdots & \tilde{A}_n \end{pmatrix} = -\det \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} = -\det A$$

by antisymmetry

$$\det \begin{pmatrix} \mu \tilde{A}_1 & \cdots & \mu \tilde{A}_n \end{pmatrix} = \mu \det \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix} = \mu \det A$$

by linearity
Ex 2/ Find \( \text{det} \begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{pmatrix} \).  

**Idea:** row-reduce to an upper triangle using only replace & swap.

\[
\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 4 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 9 & 14 & 24 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{pmatrix} = 1.9 \times 9.
\]

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**Determinants and Invertibility**

**Theorem 2:** \( \text{det} A \neq 0 \iff A \text{ invertible.} \)

**Proof:** \( (\iff) \): \( A \text{ invertible} \Rightarrow A \text{ is obtained from } I_n \text{ by EROs} \)

\[\Rightarrow \text{det} A = \text{det} I_n \cdot (-1)^{\text{num. of row swaps}}.\]

**Thm 1**

\[\Rightarrow \text{det} A = \text{det} I_n \cdot (-1) \text{ (product of scaling factors)} \]

\[\Rightarrow \text{det} A \neq 0.\]

\( (\Rightarrow) \): \( \text{det} A \neq 0 \Rightarrow \text{det}(\text{ref } A) = \text{det } A \neq 0.\)

**Thm 1**

\[\Rightarrow \text{ref } A \text{ has no row of 0s} \]

**A square**

\[\Rightarrow A \text{ invertible}.\]
Ex 3/ we know \( \text{ref} \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right) \neq I_3 \), so
\[
\det \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right) = 0.
\]

Ex 4/ Find all values of \( \alpha \) that make
\[
A = \begin{pmatrix} \alpha & 1 & 1 \\ 2 & \alpha & 1 \\ 4 & \alpha & \alpha \end{pmatrix}
\]
non-invertible ("singular").

\[
\text{Det} = 0 \Rightarrow \begin{vmatrix} \alpha & 1 & 1 \\ 2 & \alpha & 1 \\ 4 & \alpha & \alpha \end{vmatrix} = (\alpha - 1) \begin{vmatrix} \alpha & 1 \\ 4 & \alpha \end{vmatrix}
= (\alpha - 1)(\alpha^2 - 4) = (\alpha - 1)(\alpha - 2)(\alpha + 2)
\]
expansion in middle row

\[
\Rightarrow \text{values are } \alpha = 1, 2, -2.
\]

Determinants and Products

Consider the elementary matrices once more. What are their determinants?

\[
\begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = 1,
\begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} = -1,
\begin{vmatrix} 1 & \mu \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{vmatrix} = \mu
\]

Replace
Upper or lower triangular with 1's on diagonal

Swap!
Get 1 in by swapping it with rows

Scaling

Looking at the statement of Theorem 1, we see it on
be expressed as follows:

\[ \text{det}(EA) = \text{det}(E) \cdot \text{det}(A), \]

result of applying $E$ to $A$.

\[ \text{Theorem 3:} \text{ Given non-matrices } A \text{ and } B, \text{ det } AB = (\text{det } A)(\text{det } B). \]

\[ \text{Proof: Case 1: } \text{det } A = 0. \] Then $A$ isn't invertible.

If $AB$ had an inverse $C$, then $IA = (AB)C = A(BC)$

\[ \Rightarrow BC \text{ is inverse to } A, \text{ a contradiction. So } \]

$AB$ isn't invertible, and \[ \Rightarrow \text{det } AB = 0. \]

Case 2: \[ \text{det } A \neq 0. \] Then $A$ is invertible, and so

may be written as a product of elementary matrices:

\[ A = E_N \cdots E_1 (I_n) \]

\[ \Rightarrow AB = E_N \cdots E_1 B. \]

By repeated application of (Ge),

\[ \text{det } A = \text{det } E_N \cdots \text{det } E_1 \text{ det } E_1 \]

\[ \Rightarrow \text{det } AB = \text{det } E_N \cdots \text{det } E_1 \text{ det } B, \text{ so } \text{det } AB = \text{det } A \cdot \text{det } B. \]


\[ \text{Corollary:} \text{ If } A \text{ is invertible, } \text{det}(A^{-1}) = \frac{1}{\text{det } A}. \]

\[ \text{Ex 5} \]

\[ \text{det } B^{-1}A^9B = (\text{det } B)^{-9}(\text{det } A)^9 \text{ det } B = (\text{det } A)^9. \]
(What about $A+B$? In fact, we can't say anything about its determinant. It's certainly false that $\det (A+B) = \det A + \det B$; for example, take $A = \mathbf{I}_2$, $B = -\mathbf{I}_2$. Then $\det A + \det B = 1 + 1 = 2$, but $A+B$ is the zero matrix.)

Determinants and Column Vectors

I mentioned in Lecture 12 that if you define "det" to be the unique alternating multilinear normalized function on $\{\text{columns of } A\}$ then you get the same function as doing it via rows (as we've done). Since transposing takes rows to columns, this statement is equivalent to

**Theorem 4**: $\det A = \det A^T$.

**Proof**: First, $\det A \neq 0 \iff \det A^T \neq 0$, since they are both invertible (with $(A^T)^{-1} = (A^{-1})^T$) or both not.

So if they're invertible, we have $A = E_1 \cdots E_n$ and $A^T = E_n^T \cdots E_1^T$. By Thm. 3, it suffices to check $\det E_i = \det E_i^T$. But this is obvious: Swap & scale matrices are unchanged by transpose, and replace matrices here determinant 1.
Corollary (ii) Elementary column operations have the same effect on \( \det A \) as \( \text{ERO's} \).

(ii) Laplace expansion holds for columns: for each \( j \),

\[
\det A = \sum_{i=1}^{n} a_{ij} C_{ij}.
\]

Ex 6 / If

\[
\begin{vmatrix}
1 & a & b \\
g & d & e \\
h & i & f
\end{vmatrix} = 2,
\]

and

\[
\begin{vmatrix}
a-g & b-3h & c-8i \\
d & 3e & f \\
g & 3h & i
\end{vmatrix}
\]

is

\[
3 \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 3 \cdot 2 = 6.
\]

Ex 7 /

\[
\begin{vmatrix}
1 & 1 & -1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 \\
1 & 2 & 3 & \cdots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 3 & \cdots & n
\end{vmatrix}
\]

is

\[
\begin{vmatrix}
1 & 1 & -1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & 2 \\
0 & 0 & 1 & \cdots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n-1
\end{vmatrix}
\]

is

\[
= \Delta_{n-1} = \cdots = \Delta_{n-2} = \cdots = \Delta_1 = 1
\]

Determinants and Linear Independence

Recall from lectures 11 and 8 that the following conditions on an \( n \times n \) (square) matrix \( A \) are equivalent:
(C1) A is invertible \( \Rightarrow \) T is invertible
(C2) The columns of A span \( \mathbb{R}^n \) \( \Rightarrow \) T is onto
(C3) The columns of A are linearly independent \( \Rightarrow \) T is 1-1

The same goes for rows since columns of A are rows of \( A^T \).

Theorem 5: \( \det A \neq 0 \iff \) rows are lin. ind. \( \iff \) span \( \mathbb{R}^n \)
\( \iff \) columns are lin. ind. \( \iff \) span \( \mathbb{R}^n \).

Ex 8/ For what values of \( \alpha \) is \((\alpha), (1), (1)\)
\((\alpha), (1), (1)\)
a linearly independent set?

We already solved this: They're independent \( \iff \)
\( \det \begin{pmatrix} \alpha & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0 \iff \alpha \neq 1, 2, -2. \)