Lecture 17: Basis of a vector space

Let $V$ be a vector space. A finite subset $\{\vec{v}_1, \ldots, \vec{v}_n\} \subset V$ is linearly dependent if $\exists$ $c_1, \ldots, c_n \in \mathbb{R}$, not all 0, such that
\[c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n = \vec{0}.
\]
Otherwise, it is linearly independent. Linear independence is equivalent to the following property (why?): no $\vec{v}_j$ is a linear combination of the $\vec{v}_1, \ldots, \vec{v}_{j-1}$ preceding it.

**Ex 1**
In $\mathbb{P}_3$, $\{t, t^2\}$ is independent, but $\{t, t(t-2), t^2\}$ is dependent because $(-2)t + (-1)t(t-2) + (1)t^2 = 0$.

Notice that if a linear dependence relation (**) holds in $V$, and $T : V \to W$ is a linear transformation, then
\[c_1 T(\vec{v}_1) + \ldots + c_n T(\vec{v}_n) = \vec{0}
\]
holds in $W$. So we arrive at the

**Observation**: If $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ are independent, then so are $\vec{v}_1, \ldots, \vec{v}_n$.

**Ex 2**
In $C^0(\mathbb{R})$, $\{\cos(t), \sin(t), \cos(2t), \sin(2t)\}$ is independent.

OK, how would you prove that? Maybe take values of the functions at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ that give a linear transformation
\[ F: C^0(\mathbb{R}) \rightarrow \mathbb{R}^4 \]

sending

\[ \cos(t) \rightarrow (1) \]
\[ \sin(t) \rightarrow (0) \]
\[ t \cos(t) \rightarrow (0,0,0,0) \]
\[ t \sin(t) \rightarrow (1/t, 0, 0, 0) \]

You can check that the 4 vectors on the right are independent in \( \mathbb{R}^4 \). Therefore, the functions are independent in \( C^0(\mathbb{R}) \), by the Observation.

We have made a lot of use lately of the vectors \( \hat{e}_1, \ldots, \hat{e}_n \) in \( \mathbb{R}^n \). Any vector \( (c_1, \ldots, c_n) \) in \( \mathbb{R}^n \) can be written uniquely as a linear combination of them: \( \sum c_i \hat{e}_i \). But \( \hat{e}_1, \ldots, \hat{e}_n \), while especially convenient, are far from being the only set of \( n \) vectors with this property. In fact, the vectors on the right in Example 2 also have this property.

**Definition:** A finite set \( \{\hat{v}_1, \ldots, \hat{v}_n\} \subset V \) is called a basis of \( V \) if (a) it is linearly independent and (b) it spans \( V \).
Given a basis of $V$, any larger set $\{v_1, \ldots, v_n; v\}$ is dependent: since $v_1, \ldots, v_n$ span $V$, $v = q_1 v_1 + \ldots + q_n v_n$
$\Rightarrow 0 = q_1 v_1 + \ldots + q_n v_n + (-1)v'$ is a dependence relation.

**Ex 3**

1. $V = \mathbb{R}^n$ has basis $e_1, \ldots, e_n$, called the standard basis.

2. $V = \mathbb{P}_2$ has basis $\{1, t, t^2, t^3\}$.

3. $V = \mathbb{P}_2(x, y) = \text{polynomials in } x \text{ and } y \text{ of degree } \leq 2$
   has basis $\{1, x, y, xy, x^2, y^2\}$.

**Ex 4**

If $A$ is an invertible n x n matrix, its columns
$\{v_1, \ldots, v_n\}$ form a basis for $\mathbb{R}^n$. [True for rows too, since $A^T$ is invertible if $A$ is.] Why? Well, we need to check (a) and (b) in the definition:

(a): $x_1 v_1 + \ldots + x_n v_n = 0 \Rightarrow A \bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$.

So $v_1, \ldots, v_n$ is linearly independent.

(b): let $\bar{x} \in \mathbb{R}^n$, set $\bar{x} = A^{-1} \bar{v}$. Then $A \bar{x} = \bar{v}$, so $\bar{v}$ is in the span of $A$'s columns. Since $\bar{x}$ was arbitrary, $A$'s columns span $\mathbb{R}^n$. 

Our notion of bases in this course is finite—we won’t deal with infinite bases. So when does a vector space have a (finite) basis?

**Ex 5**/ \( C^0(\mathbb{R}) \) and \( P \) do not. //

**Ex 6**/ Call \( V \) finitely generated if some finite subset \( S = \{v_1, \ldots, v_n\} \) spans \( V \). In this case, some subset of \( S \) is a basis for \( V \). Why? Well, you can go through \( S \) throwing out any \( v_i \) that is a linear combination of the vectors preceding it, until this is no longer the case. To see that the resulting subset still spans \( V \), note that if (say) \( v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} \), then any linear combination \( a_1 v_1 + \cdots + a_n v_n \) can be rewritten as a linear combination

\[
(a_1 + a_2) v_1 + \cdots + (a_{k-1} + a_k) v_{k-1} + a_{k+1} v_{k+1} + \cdots + a_n v_n
\]

in which \( v_k \) is omitted. So removing \( v_k \) won’t affect the span. The new subset is linearly independent by the criterion on p. 1.

* The book calls this the “Spanning Set Theorem.”
Now let $A$ be an $m \times n$ matrix.

Consider $\text{Nul}(A) \subset \mathbb{R}^n$, the null space of $A$. Can we construct a basis?

Ex 7/ $A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \\ 2 & 4 & 3 \\ 3 & 6 & 3 \\ 3 & 6 & 3 \end{pmatrix}$ \quad \Rightarrow \quad \text{ref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$x_5, x_6$ free

$\text{Nul}(A) := \{x \mid Ax = 0\} \subset \mathbb{R}^5$

$= \left\{ \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_2 \\ 0 \end{pmatrix} \mid x_2, x_4 \in \mathbb{R} \right\}$

$= \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid x_2, x_4 \in \mathbb{R} \right\}

= \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}

\begin{align*}
\text{and in fact these 2 5-vectors form a basis.} \\
\text{The nature of the procedure in this example guarantees a basis, i.e.} \\
\text{the vectors span \text{Nul}(A); and} \\
\text{(a) there is one vector for each free variable; and if} \\
\text{this variable is } x_k, \text{ then}
\end{align*}
\[
\begin{align*}
\text{If vector has } k^{th} \text{ entry } 1 & \implies \text{the } k^{th} \text{ vector is NOT a linear combination of the others (for each } k) \\
\text{the other vectors have } k^{th} \text{ entry } 0 & \implies \text{these vectors are independent.}
\end{align*}
\]

Remark: Part of what makes this work is that row-reduction doesn't affect the null space:
\[
\text{\text{Null} } (\text{\text{ref} } (A)) = \text{\text{Null} } (A).
\]
The same is not true for the column space:
\[
\text{\text{Col} } (\text{\text{ref} } (A)) \neq \text{\text{Col} } (A).
\]

So how do we get a basis of \text{Col} (A)?

Suppose \( E \) is a product of elementary matrices which reduce \( A \) to RREF:
\[
\begin{pmatrix}
\vec{v}_1 & \ldots & \vec{v}_n \\
1 & \ldots & 1
\end{pmatrix} = \text{\text{ref} } (A) = E \cdot A = E \begin{pmatrix}
\vec{r}_1 & \ldots & \vec{r}_n \\
0 & \ldots & 0
\end{pmatrix} = \begin{pmatrix}
\vec{r}_1 & \ldots & \vec{r}_n \\
0 & \ldots & 0
\end{pmatrix}.
\]

Since \( E \) is invertible, \( \vec{v}_1, \ldots, \vec{v}_n \) have exactly the same dependence relations as \( \vec{r}_1, \ldots, \vec{r}_n \). That means that if all the \( \vec{r}_i \)'s can be written as linear combinations of some subset \( \vec{r}_{i_1}, \ldots, \vec{r}_{i_k} \) — i.e. they span a smaller subspace \( \text{\text{Col} } (\text{\text{ref} } (A)) \) — then all the \( \vec{t}_i \)'s can be written as linear combinations of \( \vec{v}_{i_1}, \ldots, \vec{v}_{i_k} \) — then span \( \text{\text{Col} } (A) \). Since the columns of
Theorem: The pivot columns of $A$ form a basis for the column space $\text{Col}(A)$.

Ex 8/ A as in Example 7, $\text{ref}(A) = \begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ pivot columns

$\Rightarrow$ a basis for $\text{Col}(A)$ is

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \\ 6 \end{pmatrix} \right\},$$

the 1st, 3rd, & 5th columns of $A$.

Upshot: We can find bases of $\text{Null}(A)$ and $\text{Col}(A)$ easily by row-reducing $A$ to RREF.