Lecture 18: Coordinates w.r.t. a basis

Coordinate Vectors Let \( V \) be a vector space, and \( B = \{ \mathbf{b}_1, \ldots, \mathbf{b}_n \} \subset V \) a basis. Take any \( \mathbf{x} \in V \).

Theorem 1: We can write \( \mathbf{x} \) as a sum \( \sum d_i \mathbf{b}_i \) in exactly one way.

Proof: Since \( B \) spans \( V \), we can write \( \mathbf{x} \) as such a sum. If \( \sum d_i \mathbf{b}_i = \mathbf{x} = \sum e_i \mathbf{b}_i \), then \( \mathbf{0} = \sum (d_i - e_i) \mathbf{b}_i \). Since \( B \) is a linearly independent set, \( d_i - e_i = 0 \) for all \( i \); that is, \( d_i = e_i \). \( \square \)

We write \( [\mathbf{x}]_B = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \) is the coordinate vector of \( \mathbf{x} \) with respect to \( B \).

Notice that the procedure of "taking the coordinate vector" gives a map
\[
[\,]_B : V \rightarrow \mathbb{R}^n
\]
\[\mathbf{x} \mapsto [\mathbf{x}]_B.\]

Theorem 2: \([\,]_B\) is a linear transformation which is also 1-1 and onto, i.e. an isomorphism.
Proof: If \( \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{b}_i \) and \( \vec{y} = \sum_{i=1}^{n} \beta_i \vec{b}_i \), then \( \vec{c} = \sum_{i=1}^{n} \gamma_i \vec{b}_i \) and \( \vec{x} + \vec{y} = \sum_{i=1}^{n} (\alpha_i + \beta_i) \vec{b}_i \).

So \( [c]_B = \left( \begin{array}{c} c_1 \\ \vdots \\ c_n \end{array} \right) = c \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) = c [\vec{x}]_B \)

and \( [x + y]_B = \left( \begin{array}{c} x_1 + \beta_1 \\ \vdots \\ x_n + \beta_n \end{array} \right) = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) + \left( \begin{array}{c} \beta_1 \\ \vdots \\ \beta_n \end{array} \right) = [\vec{x}]_B + [\vec{y}]_B \)

is linear.

If \( [\vec{x}]_B = \left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) \), then \( \vec{x} = 0 \vec{b}_1 + \cdots + 0 \vec{b}_n = \vec{0} \) (\( \Rightarrow [\vec{y}]_B \) is 1-1).

Any \( \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \) is of the form \( [x_1 \vec{b}_1 + \cdots + x_n \vec{b}_n]_B \) (\( \Rightarrow [\vec{y}]_B \) is on-to).

Linear transformations which are isomorphisms have inverses which are also linear transformations. They therefore produce an exact correspondence between linear combinations, linear independence/dependence, etc. in the two vector spaces.

**Ex 1**

\[ \begin{array}{c}
\mathbb{P}_n : (a_0 + a_1 x + \cdots + a_n x^n) \\
\rightarrow \begin{pmatrix}
\vdots \\
a_0 \\
\vdots \\
\end{pmatrix}
\end{array} \]

\[ B = \{1, x, \ldots, x^n\} \]

**Ex 2**

\[ \begin{array}{c}
\mathbb{W} : \begin{pmatrix}
\begin{array}{c}
\gamma \\
\delta \\
\end{array}
\end{pmatrix} \\
\rightarrow \begin{pmatrix}
\begin{array}{c}
\alpha \\
\beta \\
\end{array}
\end{pmatrix}
\end{array} \]

where \( W = \text{span}\{\text{cos}, \text{sin}, \text{tangent}\} \)

\[ B = \{\text{cos}, \text{sin}, \text{tangent}\} \]
Ex 3/ \[ A = m \times n \text{ matrix with } k \text{ non-pivot columns.} \]

\[ \mathbf{V} = \text{Null}(A) \subseteq \mathbb{R}^n \]

\[ \mathbf{B} = \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} \text{ the basis of } \mathbf{V} \text{ produced by our procedure in Lecture 17} \]

\[ [\cdot]_B : \mathbf{V} \rightarrow \mathbb{R}^k \]

More concretely, the plane in \( \mathbb{R}^3 \) described by \( 4x_1 - 5x_2 + 2x_3 = 0 \) is a subspace of this type, with \( A = \begin{pmatrix} 4 & -5 & 2 \end{pmatrix} \).

We have \( B = \left\{ \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5/4 \\ 1 \\ 0 \end{pmatrix} \right\} \), and \([\cdot]_B\) gives an isomorphism to \( \mathbb{R}^2 \).

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**Bases of \( \mathbb{R}^n \)**

Any basis \( B = \{ \mathbf{b}_1, \ldots, \mathbf{b}_k \} \subseteq \mathbb{R}^n \)

has an associated \( n \times k \) matrix

\[ P_B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{pmatrix} \]

Since \( B \) is linearly independent, \( \text{rank}(P_B) \) has a leading 1 in each column \( \implies k \leq n \).
Since $B$ spans IR$^n$, rank $(P_B)$ has a leading 1 in each row $\implies k = n$.

So $k = n$, and since the columns of $P_B$ span IR$^n$, $P_B$ is invertible.

What does this matter have to do with anything, though?

Ex 4/ $B = \{(5, -5), (5, -1)\}$, $[x]_B = (6, 4)$. What is $x$?

$x = 6b_1 + 4b_2 = 6(5, -5) + 4(5, -1) = (50, -36)$, done.

But notice: $x = (5, 5)^T(6, 4) = P_B[x]_B$.

Indeed, if $[x]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then by definition

$x = x_1b_1 + \cdots + x_nb_n = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P_B[x]_B$.

In fact, "$x$" is itself a coordinate vector — with respect to the standard basis $e = \{e_1, \ldots, e_n\}$:

$x = [x]_e$.

So we call $P_B$ the change-of-basis (or change-of-coordinates) matrix from $B$ to $e$. 

\[ P_B = \begin{pmatrix} 5 & 5 \\ -5 & -1 \end{pmatrix} \]
Of course, you don't need a matrix to go from \([\vec{x}]_B\) to \(\vec{x}\), as we see in the Example above. But it's great for going in the other direction.

**Ex 5**
\[B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \vec{x} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}. \text{Find } [\vec{x}]_B.\]

So: \(P_B [\vec{x}]_B = \vec{x}\), and we have 2 options: row-reduce
\[\left[ P_B \mid \vec{x} \right]\]
or use \([\vec{x}]_B = P^{-1}_B \vec{x}\). I'll go the second route:
\[
\begin{pmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -4 \\ 1 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right) = \begin{pmatrix} \frac{4}{3} \\ -1 \end{pmatrix}.
\]

**Ex 6**
Write \(p(x) = 1 + 5x - 2x^2\) with respect to the basis
\(B = \{1, x-1, (x-1)^2\}\) of \(P_2\).

Identify \(P_2\) with \(\mathbb{R}^3\) via \(a_0 + a_1 x + a_2 x^2 \leftrightarrow (a_0, a_1, a_2)\).

Then \(\vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}\), \(P_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\) \(\Rightarrow P^{-1}_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\).

\([p(x)]_B = P^{-1}_B \vec{x} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ -6 \end{pmatrix}.
\]

\([\text{Check: } 4(1) + (-1)(x-1) + (-2)(x-1)^2 = 4x - 1 - 2x^2 + 4x - 2 = 1 + 5x - 2x^2\]