Lecture 22: Some Applications

Recurrence relations

Given an n×n matrix $A$, one is sometimes interested in studying sequences of vectors $\{\mathbf{x}_k\} \subset \mathbb{R}^n$ (k ∈ \mathbb{Z} or just $\mathbb{Z}_{\geq 0}$) satisfying the first-order linear recurrence relation (or "difference equation")

\[(\dagger) \quad \mathbf{x}_{k+1} = A \mathbf{x}_k.\]

Typically, one is interested in the long-term behavior (e.g., convergence) of $\mathbf{x}_k$. Often $(\dagger)$ arises from a higher-order difference equation, as in the following example.

**Ex 1**/ Consider the set $\mathcal{S}$ of sequences $\{y_k\}_{k \in \mathbb{Z}}$ satisfying

\[y_{k+2} + a y_{k+1} + b y_k = 0 \quad (b \neq 0).\]

(i) Show that this is a subspace of

$\mathcal{S}$: vector space of all sequences $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$

of dimension 2.

(ii) Write it in the form $(\dagger)$.

(iii) Consider the linear transformation
\[ T : \mathbb{S} \rightarrow \mathbb{S} \]

\[ \{y_k\} \mapsto \{w_k\}, \text{ where } \quad w_k = y_{k+2} + ay_{k+1} + by_k. \]

\[ \mathbb{S} = \{ \text{Solutions to } (t)^3 = \ker(T) \}, \text{ which we know is a subspace.} \]

Next, look at the linear transformation

\[ R : \mathbb{S} \rightarrow \mathbb{R}^2 \]

\[ \{y_k\} \mapsto (y_0, y_1). \]

If \( y_0 = 0 = y_1 \), then using \((t)^3\) in the form \( y_{k+2} = -ay_{k+1} - by_k \)
gives \( y_{k+1} = c \), then \( y_k = 0 \), etc.; while using \((t)^3\) in the form
\[ y_k = \frac{y_{k+2} + ay_{k+1}}{-b} \]
gives \( y_{k+1} = 0 \), then \( y_k = 0 \), etc. So \( \{y_k\} \) is identically \( 0 \). This means \( R \) is \( 1-1 \).

If \( y_0 \neq y_1 \) are any values, then once again \( y_{k+2} = -ay_{k+1} - by_k \)
and \( y_{k+1} = \frac{y_{k+2} + ay_{k+1}}{-b} \) yield a solution to \((t)^3\) (compatible with \( y_0 \neq y_1 \)). So \( R \) is \( 1-1 \).

Therefore \( R \) is an \underline{isomorphism}, and \( \dim \mathbb{S} = \dim \mathbb{R}^2 = 2. \)

\[ (ii) \quad \begin{pmatrix} y_{k+1} \\ y_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y_k \\ y_{k+1} \end{pmatrix} \\
\begin{pmatrix} x_{k+1} \\ x_{k+2} \end{pmatrix} = A \cdot \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}. \]

More generally, one has the 

**Theorem:** (1) Any difference equation of the form
"order \( n \) inhomogeneous difference equation\}
\[
\begin{align*}
y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k &= z_k \\
y_0, y_1, \ldots, y_{n-1} \ 	ext{given} \\
(a_n \neq 0)
\end{align*}
\]
has a unique solution in \( S \).

(2) The set of all solutions to
\[
\begin{align*}
y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k &= 0 \\
(a_n \neq 0)
\end{align*}
\]
is an \( n \)-dimensional vector subspace of \( S \).

One way to think of this: \( S = \) space of "signal,"

\[
\text{Continuous world} \xrightarrow{\text{sample}} \text{Signal } \{y_k\} \xrightarrow{\text{filter}} \text{Filtered signal } \{z_k\}
\]

The solutions of the homogeneous difference equation are then the ones that get filtered out.

Now let's actually solve one of these.

\[
\text{Ex 2/ Find a basis of the solution space of}
\begin{align*}
y_{k+3} - 2y_{k+2} - y_{k+1} + 2y_k &= 0
\end{align*}
\]

We could say: since \( S \xrightarrow{\sim} \mathbb{R}^3 \), just set \((y_0, y_1, y_2) = (0, 0, 0), (0, 1, 0), (1, 0, 0), (y_k) \xrightarrow{(y_0, y_1, y_2)} (y_k)\)

and take the 3 corresponding sequences \((y_k)\). But a general formula for the \( y_k \)'s in this setup may be hard.

Better approach: \((1)\) If the auxiliary equation \( t^3 - 2t^2 - t + 2 = 0 \)
has a real root \( r \), then \( y_k = r^k \) is a solution. Why?
Substitute in \( r^k \) for \( y_k \):
\[
\begin{align*}
 r^{k+3} - 2r^{k+2} - r^{k+1} + 2r^k &= 0 \\
 r^3 - 2r^2 - r + 2 &= 0.
\end{align*}
\]

(2) If the auxiliary equation has distinct real roots \( r_1, r_2, r_3 \),
then \( \{r_1^k\}, \{r_2^k\}, \{r_3^k\} \) are independent in \( S \). Why?
Again, it's enough to check that the 3 values of
\[
\begin{pmatrix}
 y_0 \\
 y_1 \\
 y_2
\end{pmatrix}
\]
you get are independent, so take
\[
\begin{vmatrix}
1 & 1 & 1 \\
1 & r_1 & r_2 \\
1 & r_1^2 & r_2^2
\end{vmatrix} = (r_2 - r_1)(r_3 - r_1) - (r_3 - r_1)(r_2 - r_1) = 0.
\]
(This is nonzero since \( r_1, r_2, r_3 \) are distinct.)

Solution: Solve the auxiliary equation \( (t+1)(t-1)(t-2) = 0 \)
\[
\Rightarrow \{(-1)^k\}, \{1^k\}, \{2^k\} \text{ are the desired basis.}
\]

**Markov chains**

In probability theory, the knowledge of previous experimental outcomes can influence predictions for future experiments — this is called **conditional probability**.

If you repeatedly carry out such an experiment or measurement, then you get a sequence of probability distributions and can ask where they go in the long run.
here the probability distribution will be represented by a
state vector whose entries sum to 1, say
\[ \mathbf{x}_k = \begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix} \]
where \( P_k(E) \) is the probability of event \( E \) occurring at the \( k \)-th step.

We have a matrix of conditional probabilities
\[
P = \begin{pmatrix} P(A|A) & P(A|B) \\ P(B|A) & P(B|B) \end{pmatrix}
\]
where \( P(E|F) \) is the probability event \( E \) occurs on next step when \( F \) has occurred at current step

in which the columns sum to 1 —
such a matrix is called stochastic. (This ensures that
the entries of \( \mathbf{x}_k \) sum to 1 when \( \mathbf{x}_k \)'s do:
\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \mathbf{x}_k = (1, 1) \cdot \begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix} = (1, 1) \mathbf{x}_k = 1.
\]

So then
\[
\begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix} = \begin{pmatrix} P(A|A) & P(A|B) \\ P(B|A) & P(B|B) \end{pmatrix} \begin{pmatrix} P_k(A) \\ P_k(B) \end{pmatrix}
\]
\[
\mathbf{x}_{k+1} = P \cdot \mathbf{x}_k,
\]
and the sequence
\[ \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \]
is called a Markov chain.

\textbf{Ex 3}
(Winter in MD) / There are 3 kinds of weather:
\[ W = \text{warm \, \text{e.g. a warm day}} \]
\[ T = \text{tornadoes} \]
\[ C = \text{cold/clear} \]

So our stochastic matrix is
\[
P = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{2} & 0 & 0 \\
\frac{1}{4} & 1 & \frac{3}{4}
\end{pmatrix}
\]

and today is cold (sort of). Then what is the probability of a tornado the day after tomorrow?

\[
P^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \rightarrow 12.5\% \text{ chance.}
\]

What is the probability of a tornado on an "average day in the long run"?

Rather than try to compute \( P^N \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) for \( N \) big, what we're really after is a vector \( \mathbf{x} \) with the property that

\[
P \mathbf{x} = \mathbf{x},
\]

i.e. a steady-state vector. (There is a unique such vector if some power of \( P \) — in this case, \( P^3 \) — has all positive entries.) To find \( \mathbf{x} \), write
\[ \hat{0} = P \hat{x} - \hat{x} = (P - \mathbb{I}_n) \hat{x} . \]

That is, we are after the null space of

\[ P - \mathbb{I}_3 = \begin{bmatrix}
-3/4 & 0 & 1/4 \\
1/2 & -1 & 0 \\
1/4 & 1 & -1/4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2 & 0 \\
0 & 3/2 & 1/4 \\
0 & 3/2 & -1/4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -1/3 \\
0 & 1 & -1/2 \\
0 & 0 & 0
\end{bmatrix} . \]

So a solution is \( \begin{pmatrix} 1/3 \\ 1/6 \\ 1 \end{pmatrix} \) — but the sum of its entries is 3/2. To get a state vector, divide by this:

\[ \hat{x} = \frac{2}{3} \begin{pmatrix} 1/3 \\ 1/6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 1/9 \\ 2/3 \end{pmatrix} \quad \text{and Answer: } \frac{1}{3} . \]

More generally, our state vectors might represent the distribution of the population or vote (our different locations resp. candidates) — see the examples in the text.