Lecture 4: Matrix Equations

Multiplying a matrix by a (column) vector: 2 approaches

1. **Row-by-column** (more computationally efficient)

   \[
   \begin{pmatrix}
   a_1 & b_1 & c_1 \\
   a_2 & b_2 & c_2 \\
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y \\
   \end{pmatrix}
   =
   \begin{pmatrix}
   a_1x + b_1y + c_1z \\
   a_2x + b_2y + c_2z \\
   \end{pmatrix}.
   \]

   Notice that this is:

   \[
   =
   \left(\begin{array}{c}
   a_1x \\
   a_2x \\
   \end{array}\right) +
   \left(\begin{array}{c}
   b_1y \\
   b_2y \\
   \end{array}\right) +
   \left(\begin{array}{c}
   c_1z \\
   c_2z \\
   \end{array}\right),
   \]

   which leads to...

2. **Linear combinations** (more conceptual)

   Writing \( A = \begin{pmatrix}
   v_1 & v_2 & \ldots & v_n \\
   \end{pmatrix} \), \( \vec{x} = \begin{pmatrix}
   x_1 \\
   x_2 \\
   \vdots \\
   x_n \\
   \end{pmatrix} \),

   \[
   A \vec{x} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n.
   \]
As an example of the second approach's utility, we can use it to check the

\[ \text{Linearity property: } A(c\hat{u} + d\hat{w}) = cA\hat{u} + dA\hat{w} \]

By (\ref{eq:linear}),

\[
\begin{align*}
\text{LHS} & = (cu_i + dw_i)\hat{v}^i + \ldots + (cn_i + dw_n)\hat{v}^n \\
& = c(u_i\hat{v}^i + \ldots + u_n\hat{v}^n) + d(w_i\hat{v}^i + \ldots + w_n\hat{v}^n) \\
& = \text{RHS}.
\end{align*}
\]

More importantly, we see that

**Statement 1:** \( A\hat{x} = \hat{b} \) has a solution in \( \mathbb{R} \) 
(i.e. is consistent)

**is equivalent to**

**Statement 2:** \( \hat{b} \in \text{Span} \{ \text{columns of } A \} \)

The link is (\ref{eq:link}): \( \hat{b} = A\hat{x} = x_1\hat{v}_1 + x_2\hat{v}_2 + \ldots + x_n\hat{v}_n \) says “you can choose \( x_1, \ldots, x_n \) so that \( \hat{b} \) is a linear combination of \( \hat{v}_1, \ldots, \hat{v}_n \).”

So to check Statement 2, you just row-reduce \([A \mid \hat{b}]\).
Here is another set of equivalent statements:

**Assertion (A):** \( \text{Span} \{ \text{columns of } A \} = \{ \text{all of } 1 \} \mathbb{R}^m \)

\( \iff \) (clear from above)

**Assertion (B):** \( A \vec{x} = \vec{b} \) is consistent for any \( \vec{b} \)

\( \iff \) why?

**Assertion (C):** \( \text{ref}(A) \) has no rows of all zeroes.

First,

- \( [A \mid \vec{b}] \overset{\text{row equiv}}{\sim} [\text{ref}(A) \mid \vec{c}] \) for some vector \( \vec{c} \)

  (apply the sequence of row operations that put \( A \) in RREF to the augmented matrix)

- we can choose \( \vec{b} \) so that \( \vec{c} \) is any vector

  (because row operations are reversible)

\[(C) \Rightarrow (B): \text{ If (C) holds, then } [\text{ref}(A) \mid \vec{c}]

has a leading '1' in every row in the "ref(A)" part, hence is in RREF itself and = \text{ref}[A \mid \vec{b}]. Since the leading '1's occur to the left of \( \vec{c} \), \( \vec{c} \) is not a pivot column and the system is consistent (regardless of \( \vec{b} \)). So (B) holds.\]
(B) $\Rightarrow$ (C): If (C) fails, choose $\tilde{b}$ so that $\tilde{c}$ has a nonzero last entry. Since the last row of $\text{ref}(A)$ is all 0’s, $\tilde{c}$ is a pivot column (for this choice of $\tilde{b}$), and so (B) fails.

Ex 1/ For which $\tilde{b}$ is $\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} x = \tilde{b}$ solvable?

(Equivalent question: determine $\text{span}\{\begin{pmatrix} 3 \\ -9 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\}$.)

\[
\begin{bmatrix}
3 & -1 & \mid & b_1 \\
-9 & 3 & \mid & b_2
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & -\frac{1}{3} & \mid & \frac{b_1}{3} \\
0 & 3 & \mid & b_2
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & -\frac{1}{3} & \mid & \frac{b_1}{3} \\
0 & 0 & \mid & b_2 + 3b_1
\end{bmatrix}
\]

so we must have $b_2 = -3b_1$, i.e. $\tilde{b}$ must be a scalar multiple of the vector $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

Ex 2/ Do the columns of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ span $\mathbb{R}^3$?

(Equivalent question: does $\text{ref}(A)$ have no rows of “all 0’s”?)

Row reduce:
\[
A \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & 16 & 24 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{ref}(A)
\]

Answer: NO.
Ex 3/ Suppose $A = 4 \times 4$ matrix, and $\mathbf{b} \in \mathbb{R}^4$ are such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Must columns of $A$ span $\mathbb{R}^4$?

Consider the augmented matrix $[A | \mathbf{b}]$ and its ref, which must have:

- no non-pivot columns in the "[ ... ]" part of the augmented matrix (for uniqueness)
- free variables

So all columns of $A$ are pivot $\implies$ all rows of ref

AND $A$ is $4 \times 4$ $\implies$ have a leading '1'

$\implies$ no rows of zeroes $\implies$ columns span $\mathbb{R}^4$.

Notice that if $A$ was instead $5 \times 4$, this argument breaks down and columns need not span $\mathbb{R}^5$ (in fact, they can't).