**Ex 1**/ Consider the vectors \( \mathbf{v}_1 = (4, 4, 7) \), \( \mathbf{v}_2 = (2, 5, 8) \), and \( \mathbf{v}_3 = (3, 6, 9) \) in \( \mathbb{R}^3 \). Do they span a line, a plane, or all of space? That is, do they look like

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
1 \\
4 \\
7
\end{pmatrix}x + \begin{pmatrix}
2 \\
5 \\
8
\end{pmatrix}y + \begin{pmatrix}
3 \\
6 \\
9
\end{pmatrix}z = \mathbf{b} \text{ consistent?}
\]

To find out, we could ask "for what kinds of \( \mathbf{b} \) is \( \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
1 \\
4 \\
7
\end{pmatrix}x + \begin{pmatrix}
2 \\
5 \\
8
\end{pmatrix}y + \begin{pmatrix}
3 \\
6 \\
9
\end{pmatrix}z = \mathbf{b} \text{ consistent}?
\]

Writing this as an augmented matrix and row-reducing gives

\[
\begin{pmatrix}
1 & 2 & 3 & | & 1 \\
4 & 5 & 6 & | & 0 \\
7 & 8 & 9 & | & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & | & 1 \\
0 & -3 & -6 & | & 0 \\
0 & -6 & -12 & | & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & | & 0 \\
0 & 1 & 2 & | & 0 \\
0 & 0 & 0 & | & 0
\end{pmatrix}
\]

So our vectors span the plane described by \( b_1 - 2b_2 + b_3 = 0 \). We can also write this as the span of \( \begin{pmatrix}
2 \\
1 \\
0
\end{pmatrix} \) and \( \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \) (why?).
Definition: Let \( \vec{v}_1, \ldots, \vec{v}_n \) be a finite sequence of vectors in \( \mathbb{R}^m \). A linear dependence relation among these vectors is an equation

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0}
\]

in which \( c_1, \ldots, c_n \) are real numbers that are not all zero. A sequence of vectors with such a relation is called linearly dependent.

Ex 1 (cont’d) /

Let’s show that \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly dependent: we must solve

the equation

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Row-reducing the augmented matrix yields

\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\( x = 2 \)

\( y = -2z \).

Picking any nonzero \( z \) will do — say, \( z = 1 \). Then we have

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
2 \\
-2 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\rightarrow
1 \begin{pmatrix}
1 \\
4 \\
7
\end{pmatrix}
-2 \begin{pmatrix}
2 \\
5 \\
8
\end{pmatrix}
+ 1 \begin{pmatrix}
3 \\
6 \\
9
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]
Ex 2/ The vectors $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

...how span all of $\mathbb{R}^3$, and clearly also have no linear dependence relation. How should we phrase the latter property?

\[ \text{Definition: A sequence of vectors } \vec{v}_1, \ldots, \vec{v}_n \text{ in } \mathbb{R}^m \text{ is linearly independent if it is not linearly dependent: that is, if the only solution to} \]

\[ x_1 \vec{v}_1 + x_2 \vec{v}_2 + \ldots + x_n \vec{v}_n = \vec{0} \]

\[ \text{is } \vec{x} = \vec{0}. \text{ So: to check that } \vec{v}_1, \ldots, \vec{v}_n \text{ are independent, you must show the implication} \]

\[ c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n = \vec{0} \quad \text{implies} \quad c_1 = \ldots = c_n = 0. \]

$\left( c_1, \ldots, c_n \in \mathbb{R} \right)$

Ex 2 (cont'd) /

Suppose $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Then \( \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = c_3 = 0. \text{ Therefore} \]

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ \text{ are linearly independent.} \]
Two equivalent conditions on columns \( \vec{v}_1, \ldots, \vec{v}_n \) of an \( m \times n \) matrix \( A \):

(I) The columns of \( A \) are linearly independent.

(II) All columns of \( \text{ref}(A) \) contain a leading ‘1’. (i.e., all columns of \( A \) are pivot columns)

Check: \( (II) \Rightarrow (I) \): Suppose \( x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n = \vec{0} \).

Since \( (II) \) holds, row-reducing \( \begin{bmatrix} A & | & \vec{0} \end{bmatrix} \)

yields \( \begin{bmatrix} 1 & \cdots & 1 & \mid & 0 \\ \vdots & \ddots & \vdots & | & \vdots \\ 0 & \cdots & 0 & | & 0 \end{bmatrix} \Rightarrow \) only solution is \( \begin{cases} x_1 = 0 \\ \vdots \\ x_n = 0 \end{cases} \).

\( (II) \Rightarrow (I) \): If \( \text{ref} \left[ A \mid \vec{0} \right] = \begin{bmatrix} 1 & \cdots & 1 & | & 0 \\ \vdots & \ddots & \vdots & | & \vdots \\ 0 & \cdots & 0 & | & 0 \end{bmatrix} \)

has a non-pivot column (say, the \( i \)th),

then there exists a solution to \( x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n = \vec{0} \)

(i.e., \( A \vec{x} = \vec{0} \)) in which \( x_i \) can be anything we want (in particular, nonzero). This gives a linear dependence relation.
Ex 3/ Are the vectors \((\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix})\) linearly independent in \(\mathbb{R}^4\)?

Row-reduce:
\[
\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -3 & \frac{4}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \text{NO!}
\]

To find a linear dependency, take for example \(x_3 = 3, x_4 = 0\), so \(x_1 = 1\) and \(x_2 = -2\): 
\[
\begin{pmatrix} 1 \\ -2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -1 \\ -\frac{5}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

Notice that (as above, \(A = \text{m} \times \text{n}\) matrix):

1. If \(m < n\) (more than \(n\) vectors in \(\mathbb{R}^m\)), then the columns of \(A\) are never independent.
   (Because there are only \(m\) rows to have leading ‘1’s, and there are more columns than that.)

2. If \(m = n\) (square matrix), then columns are independent \(\iff\) \(\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\)

\(\iff\) columns span \(\mathbb{R}^m\).
(The last equivalence follows from Lecture 4, since the m×m identity matrix \((1, \ldots, 0)\) has no rows of "all 0" at the bottom.)

(3) If a sequence of vectors \(\vec{v}_1, \ldots, \vec{v}_n\) contains \(\vec{0}\), then that sequence is dependent.
   (Say \(\vec{v}_1 = \vec{0}\). Then \(1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \ldots + 0 \cdot \vec{v}_n = \vec{0}\) is a linear dependency.)

(4) A sequence is linearly dependent \(\iff\) at least one of the \(\vec{v}_i\) is a linear combination of the others.
   (If \(c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n = \vec{0}\), with some \(c_k \neq 0\), then by rescaling we can assume \(c_k = 1\). Then \(\vec{v}_k = \frac{-c_1}{c_k} \vec{v}_1 - \ldots - \frac{c_{k-1}}{c_k} \vec{v}_{k-1} - \frac{c_{k+1}}{c_k} \vec{v}_{k+1} - \ldots - \frac{c_n}{c_k} \vec{v}_n\).
   The converse is also clear from this.)

**Warning:** Consider \(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\).

This is linearly dependent, and \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\) are each linear combinations of the others. But \(\vec{v}_4\) isn't. So (4) means exactly what it says ("at least one", not "every").
Finally, why do I not speak simply of the set of vectors $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ being (in)dependent? I say "finite sequence" (the book "indexed set"). This is because repetitions matter, and a plain "set" doesn't remember repetitions. If, for example, $A = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_3 \end{pmatrix}$ and $\tilde{v}_1 = \tilde{v}_2 (\neq \emptyset)$, but $\tilde{v}_2$ is not a multiple of $\tilde{v}_1$, the set of column vectors is just $\{\tilde{v}_2, \tilde{v}_3\}$. But (i) $\tilde{v}_2, \tilde{v}_3$ is independent while

(ii) $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ is dependent b/c of the relation

$$1 \tilde{v}_1 + (-1) \tilde{v}_2 + 0 \tilde{v}_3 = 0.$$

In this situation, the theorem would all be wrong if we considered columns of $A$ to be independent: we must keep track of repetitions, and it is (ii) that matters.

So: to be absolutely clear — a sequence of vectors $\tilde{v}_1, \ldots, \tilde{v}_n$ in which the same vector appears twice is ALWAYS dependent.