Lecture 8: Matrix of a linear transformation

Consider the examples

1. \( T: \mathbb{R}^2 \to \mathbb{R}^2 \), \( T(y) = (x) \) "flip"

and

2. \( T: \mathbb{R}^3 \to \mathbb{R}^3 \), \( T(z) = (0) \) "projection"

of linear transformations. These are given by matrices

\[
(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

For both of these, we can ask two questions:

Q1 ("Existence"): Is every \( b \in \mathbb{R}^m \) (or \( \mathbb{R}^3 \) resp.) in the image of \( T \)? Recall that this is the same thing as existence of \( \mathbf{x} \) such that \( T(\mathbf{x}) = A \mathbf{x} = b \). So, YES for (1), NO for (2).

Q2 ("Uniqueness"): Given a \( b \) for which \( T(\mathbf{x}) = b \) has a solution, is this solution unique?

Again: YES for (1), NO for (2).

We'll develop a systematic way to check this shortly.
First, we ask whether every linear transformation is a matrix transformation? If we write
\[ \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n, \]
then using linearity of \( T : \mathbb{R}^n \to \mathbb{R}^m \) gives
\[ T(\vec{y}) = T(x_1 \vec{e}_1 + \ldots + x_n \vec{e}_n) = x_1 T(\vec{e}_1) + \ldots + x_n T(\vec{e}_n) \]
\[ = \begin{pmatrix} \uparrow & \uparrow & \vdots & \uparrow \\ T(\vec{e}_1) & \ldots & \ldots & T(\vec{e}_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \]
\[ = : A \]
\[ = A \vec{x}. \]

So we not only see that the answer is YES — we get a formula!

Ex /
Suppose \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) has \( T(\vec{e}_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \),
\( T(\vec{e}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \), and \( T(\vec{e}_3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). What is its matrix?

\[ A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}. \]
Ex. Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation by angle $\theta$ about the $x$-axis. Find $A$.

\[ T(\hat{e}_1) = \hat{e}_1 \]
\[ T(\hat{e}_2) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \]
\[ T(\hat{e}_3) = \begin{pmatrix} 0 \\ -\sin \theta \end{pmatrix} \]

So
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}
\]

Ex. Find the matrix of the "shear" transformation depicted:

\[
A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\]

(doesn't affect the $xy$-plane)
"Onto" and "1-1" abbreviation for "one-to-one"

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, domain $\mathbb{R}^n$, codomain $\mathbb{R}^m$ with matrix $A$.

Definition: (a) $T$ is onto if the range (image) of $T$ equals $\mathbb{R}^m$.
(b) $T$ is 1-1 if no two distinct $v, \overrightarrow{v} \in \mathbb{R}^n$ are sent (by $T$) to the same vector in $\mathbb{R}^m$.

Theorem 1: The following are equivalent:

1. $T$ onto (any $\overrightarrow{b} \in \mathbb{R}^m$ can be written $T(\overrightarrow{x})$)
2. $A \overrightarrow{x} = \overrightarrow{b}$ is consistent for all $\overrightarrow{b} \in \mathbb{R}^m$
3. Columns of $A$ span $\mathbb{R}^m$
4. $\text{rref}(A)$ has no rows of all zeros (leading '1' in every row)

Clearly this pertains to the "Existence" question above.

The uniqueness of a solution is determined by whether $T$ is 1-1, about which we have:
Theorem 2: The following are equivalent:

51' T is 1-1

52' $A\vec{x} = \vec{0}$ has only the zero solution

53' The columns of $A$ are linearly independent

54' $\text{ref}(A)$ has a leading '1' in every column.

Why are 51' and 52' equivalent? Clearly if T is 1-1, then $A\vec{x} = \vec{0}$ can't have more than the 0 solution.

If T is not 1-1, then there exist $\vec{v}_1, \vec{v}_2$ distinct with $T\vec{v}_1 = T\vec{v}_2$, which implies (by linearity of T) that $\vec{0} = T\vec{v}_1 - T\vec{v}_2 = T(\vec{v}_1 - \vec{v}_2)$, where $\vec{v}_1 - \vec{v}_2$ is different from $\vec{0}$.

Ex/ Consider the "flip" and "projection" examples on the first page. The flip had $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and is 1-1 and onto. The projection had $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and is neither 1-1 nor onto. But if we took its codomain to be instead $\mathbb{R}^2$, then $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and it is onto. (Just remember that "onto" means "onto the entire codomain".)
Ex: Say \( T: \mathbb{R}^4 \to \mathbb{R}^3 \) has matrix \( A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ -1 & 0 & -6 & -2 \end{pmatrix} \).

Is it 1-1? Onto?

Row-reduce: \( A \to \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \) → \( \begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \). No rows of 0s ⇒ \( T \) onto.

Has a non-pivot column ⇒ \( T \) not 1-1.

Q3: Can a linear \( T: \mathbb{R}^4 \to \mathbb{R}^3 \) ever be 1-1?

Can a linear \( T: \mathbb{R}^3 \to \mathbb{R}^4 \) ever be onto?

No in both cases, using \( \text{S4}/\text{S4}^* \) in the two theorems.