VECTORS

1

In order to use Calculus to treat curvilinear motion (e.g. Kepler's Laws) and to study functions of several variables, we will need to be comfortable working with vectors and matrices.

Today we review vectors in $\mathbb{R}^n$, which is to say ordered n-tuples of real numbers

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n .$$

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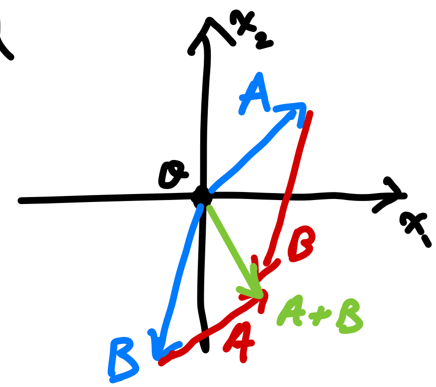
follow from the same property for real #s

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

§ 1. Vector addition

Define $A+B := \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix}$.

- may be visualized by "adding arrows head to tail"

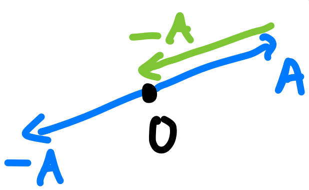


- Commutativity ($A+B = B+A$) corresponds to the fact that the vectors A, B give the sides of a parallelogram
- associativity ($A+(B+C) = (A+B)+C$)

§2. Scalar multiplication

Set $cA := \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}$ $(c \in \mathbb{R})$

- may be visualized by dilation of arrows (and reversing their direction if $c < 0$), e.g. $-A + A = 0$



- associativity: $c(dA) = (cd)A$
- distributivity: $c(A+B) = cA + cB$, $(c+d)A = cA + dA$

§1. Vector addition

Define $\underset{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}{A} + \underset{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}{B} := \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$.

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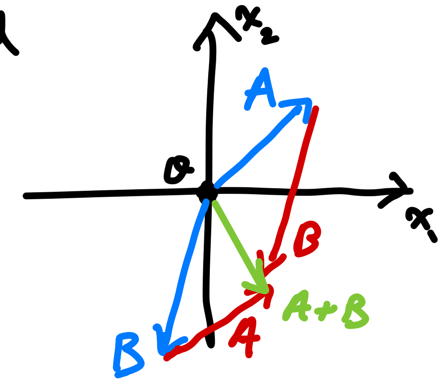

- Commutativity $(A+B = B+A)$ corresponds to the fact that the vectors A, B give the sides of a parallelogram

- associativity $(A+(B+C) = (A+B)+C)$

- $0 + A = A$, where $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

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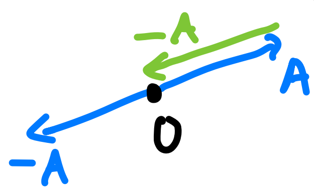
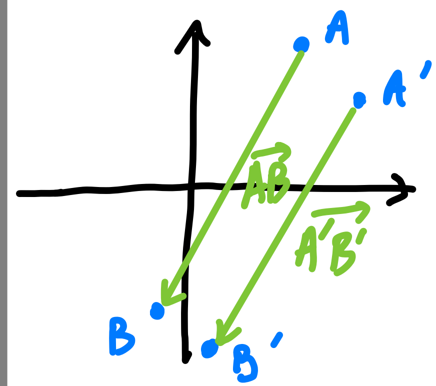


- associativity : $c(dA) = (cd)A$
- distributivity : $c(A+B) = cA + cB$, $(c+d)A = cA + dA$
- linear combinations : $cA + dB$, e.g. if $c = 1$ & $d = -1$ we get "$A - B$"

Remark: To be a bit more precise about arrows vs. points in real n-space, one would denote points by A, B, etc. and the arrow from A to B by $\overrightarrow{AB}$.



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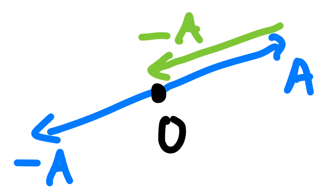

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Definition: Two vectors A & B are parallel [resp. have the $\left\{\begin{array}{l}\text{same direction} \\ \text{opposite direction}\end{array}\right.$] if $A = cB$ for some $c \in \mathbb{R} \setminus \{0\}$ [resp. $c \in \mathbb{R}_+$, $c \in \mathbb{R}_-$].

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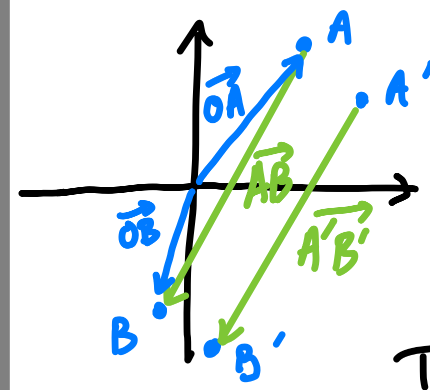
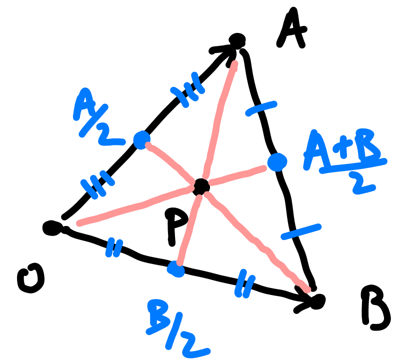


If O denotes the origin, usually one would then write $\overrightarrow{OA}$ as "A", $\overrightarrow{OB}$ as "B". The picture then demonstrates that $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \implies$

$A + \overrightarrow{AB} = B \implies \overrightarrow{AB} = B - A \in \mathbb{R}^n$

So if $A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ & $B = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ then $\overrightarrow{AB} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$. Notice that if $A' = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $B' = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, then $\overrightarrow{A'B'} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$. Hence $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are the same element of $\mathbb{R}^2$: all that matters is the ordered 2-tuple of real numbers.

Example: The medians of a triangle are concurrent.



Clearly $P := \frac{A+B}{3}$ is on the line thru O and $\frac{A+B}{2}$.

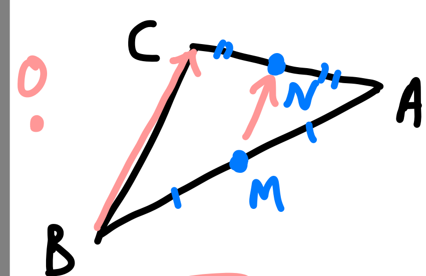
It also lies on the line thru $B/2$ and A, since

$$A + \tfrac{2}{3}\left(\tfrac{B}{2} - A\right) = \tfrac{1}{3}A + \tfrac{1}{3}B = P.$$

Finally, it lies on the 3rd median:

$$B + \tfrac{2}{3}\left(\tfrac{A}{2} - B\right) = \tfrac{1}{3}A + \tfrac{1}{3}B = P. \qquad \square$$

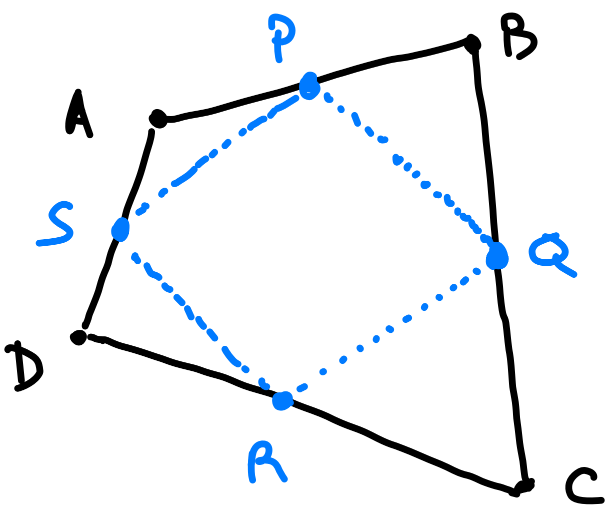
Example: Given ΔABC, let M & N be the midpoints shown. Show $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$.



"N" means $\overrightarrow{ON}$, etc.

$$\begin{aligned}\overrightarrow{MN} &= N - M \\ &= \frac{A+C}{2} - \frac{A+B}{2} \\ &= \frac{C-B}{2} \\ &= \frac{1}{2}\overrightarrow{BC}. \qquad \square\end{aligned}$$

Exercise Let $ABCD$ be an arbitrary quadrilateral. Let P, Q, R, S be the midpoints of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, & $\overrightarrow{DA}$ resp. Show that $PQRS$ is a parallelogram.



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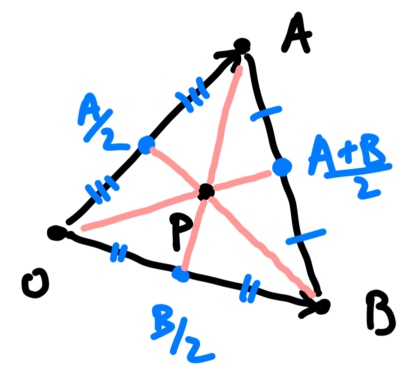

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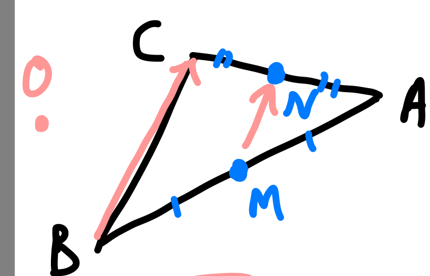

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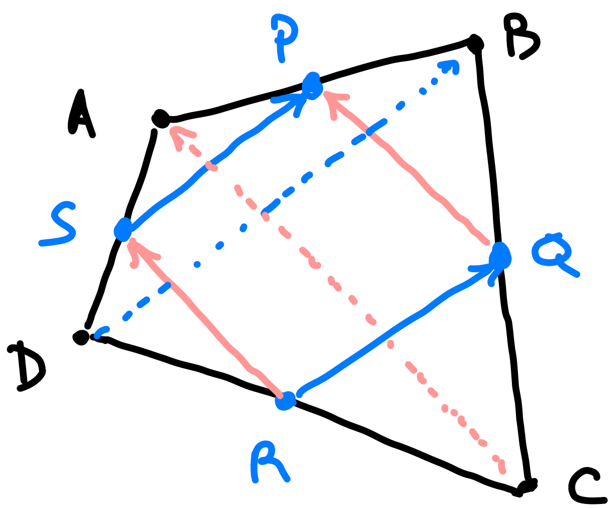
$\boxed{\text{Exercise}}$ Let ABCD be an arbitrary quadrilateral. Let P, Q, R, S be the midpoints of $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, & $\overrightarrow{DA}$ resp. Use vectors to show PQRS is a parallelogram.
[Hint. Use the last example.]



$$\begin{cases} \overrightarrow{RQ} = \frac{1}{2}\overrightarrow{DB} = \overrightarrow{SP} \\ \overrightarrow{QP} = \frac{1}{2}\overrightarrow{CA} = \overrightarrow{RS} \end{cases}$$

by the last Example.

§3. Dot product $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$$A \cdot B := \sum_{k=1}^{n} a_k b_k = a_1 b_1 + \cdots + a_n b_n.$$

- Commutative: $A \cdot B = B \cdot A$
- distributive: $A \cdot (B + C) = A \cdot B + A \cdot C$
- homogeneous: $c(A \cdot B) = (cA) \cdot B = A \cdot (cB)$
- positive-definite: $A \cdot A > 0$ unless $A = 0$ ($\Rightarrow A \cdot A = 0$)

(why? $A\,A = \sum_{k=1}^{n} a_k^2$)

- Cauchy-Schwarz inequality:

$$|A \cdot B| \leq \|A\|\|B\|.$$

§3. Dot product

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- commutative: $A \cdot B = B \cdot A$
- distributive: $A \cdot (B+C) = A \cdot B + A \cdot C$
- homogeneous: $c(A \cdot B) = (cA) \cdot B = A \cdot (cB)$
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(why? $A \cdot A = \sum_{k=1}^{n} a_k^2$)

Definition: The length (or norm) of A is $\|A\| := \sqrt{A \cdot A}$.

Note: $\|cA\|^2 = cA \cdot cA = c^2(A \cdot A) = c^2\|A\|^2$

$\Rightarrow \|cA\| = |c|\,\|A\|$.

This also gives a notion of distance

$$d(A, B) = \|\vec{AB}\| = \|B - A\|.$$

- Cauchy–Schwarz inequality:

$$|A\cdot B| \le \|A\|\|B\|$$

Proof: If $B=0$ it's obvious. So assume $B\neq 0$, and write

$$0 \le \left\| \|B\| A - \frac{A\cdot B}{\|B\|} B \right\|^2$$

$$= \left(\|B\| A - \frac{A\cdot B}{\|B\|} B\right)\cdot\left(\|B\| A - \frac{A\cdot B}{\|B\|} B\right)$$

$$= \|B\|^2 A\cdot A - \frac{A\cdot B}{\cancel{\|B\|}}\cancel{\|B\|}\, \overset{A\cdot B}{B\cdot A} - \frac{A\cdot B}{\cancel{\|B\|}}\cancel{\|B\|}\, A\cdot B + \frac{(A\cdot B)^2}{\cancel{\|B\|^2}} \cancel{B\cdot B}$$

$$= \|B\|^2\|A\|^2 - 2(A\cdot B)^2 + (A\cdot B)^2$$

$$= \|B\|^2\|A\|^2 - (A\cdot B)^2$$

$$\Rightarrow \|B\|^2\|A\|^2 \ge (A\cdot B)^2$$

$$\Rightarrow \|B\|\|A\| \ge (A\cdot B).$$ $\square$

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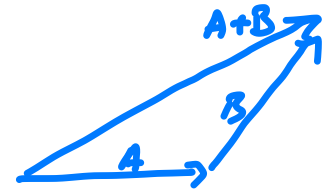
- **Triangle inequality:**



$$\|A+B\| \le \|A\| + \|B\|$$

Proof: $\|A+B\|^2 = (A+B)\cdot(A+B)$

$$= A\cdot A + B\cdot B + A\cdot B + B\cdot A$$

$$= \|A\|^2 + \|B\|^2 + 2A\cdot B$$

by Cauchy-Schwarz

$$\le \|A\|^2 + \|B\|^2 + 2\|A\|\,\|B\|$$

$$= (\|A\| + \|B\|)^2.$$

Take $\sqrt{\ }$'s to conclude. $\square$

- Cauchy-Schwarz inequality:

$$|A \cdot B| \le \|A\|\|B\|$$

... this motivates the

Definition: $A, B \in \mathbb{R}^n$ are called orthogonal (or perpendicular) if $A \cdot B = 0$. (We write $A \perp B$.)

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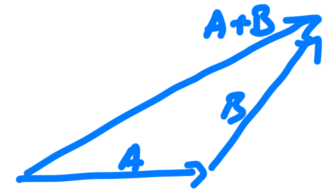

Proof:
$$
\begin{aligned}
\|A+B\|^2 &= (A+B)\cdot(A+B) \\
&= A\cdot A + B\cdot B + A\cdot B + B\cdot A \\
&= \|A\|^2 + \|B\|^2 + 2A\cdot B \\
&\overset{\text{by Cauchy-Schwarz}}{\le} \|A\|^2 + \|B\|^2 + 2\|A\|\|B\| \\
&= (\|A\| + \|B\|)^2 .
\end{aligned}
$$

Take $\sqrt{\ }$'s to conclude. $\square$

- Orthogonality: If $C = A+B$, the last proof gives

$$\|C\|^2 = \|A\|^2 + \|B\|^2 + 2A\cdot B .$$

If A & B are perpendicular we should have $\|C\|^2 = \|A\|^2 + \|B\|^2$ by the Pythagorean Theorem.

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Definition: $A, B \in \mathbb{R}^n$ are called orthogonal (or perpendicular) if $A\cdot B = 0$. (We write $A \perp B$.)

Example: $A = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $B = \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix}$

$A\cdot A = 30 \implies \|A\| = \sqrt{30}$

$B\cdot B = 101 \implies \|B\| = \sqrt{101}$

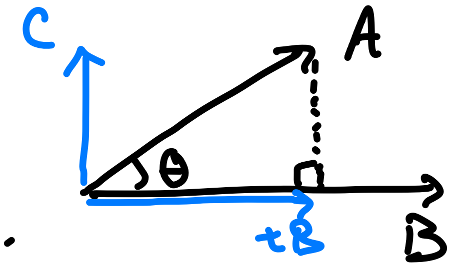
$A\cdot B = 0 \implies A \perp B$

We can produce unit vectors in the directions of A & B by $\frac{A}{\|A\|} = \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ etc.

(works b/c $\frac{A}{\|A\|}\cdot\frac{A}{\|A\|} = \frac{A\cdot A}{\|A\|^2} = \frac{\|A\|^2}{\|A\|^2} = 1$)

The distance $d(A,B) = \|A-B\| = \left\|\begin{pmatrix} 9 \\ -1 \\ -7 \end{pmatrix}\right\| = \sqrt{131}$. $\square$

- Projections:



We'd like to calculate the projection tB of A onto B, as shown.

To do this, write $A = C + tB$, where $C\cdot B = 0$. This gives

$$A\cdot B = C\cdot B + tB\cdot B = t\,B\cdot B \implies$$

$$t = \frac{A\cdot B}{B\cdot B} \implies \underbrace{tB}_{=\,\mathrm{proj}_B A} = \underbrace{\left(A\cdot\frac{B}{\|B\|}\right)}_{\text{magnitude}}\underbrace{\frac{B}{\|B\|}}_{\substack{\text{direction} \\ \text{(unit vector)}}}.$$

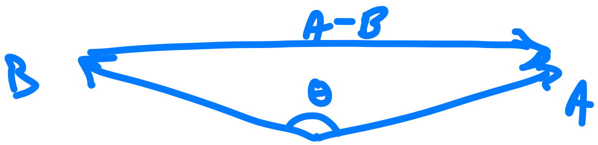
Moreover, we get

$$\cos\theta = \frac{t\|B\|}{\|A\|} = \frac{A\cdot B\,\cancel{\|B\|}}{\|B\|^{\cancel{2}}\,\|A\|} = \frac{A}{\|A\|}\cdot\frac{B}{\|B\|}$$

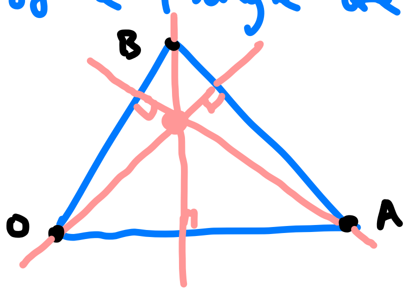
which makes sense by Cauchy-Schwarz ($\Rightarrow |\mathrm{RHS}| \le 1$).

Exercise Prove the law of cosines

$$\|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\|\cos\theta$$



Exercise Show that the perpendicular bisectors of a triangle are concurrent.



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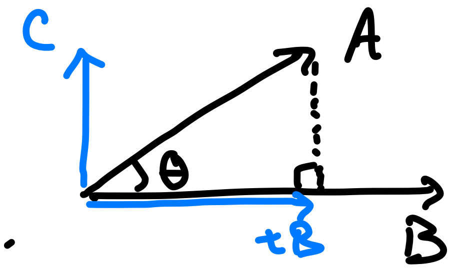

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Example: $A = \begin{pmatrix}1\\2\\1\end{pmatrix}$, $B = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\rightsquigarrow$

$$\cos\theta = \frac{A\cdot B}{\|A\|\|B\|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{6}. \quad \square$$

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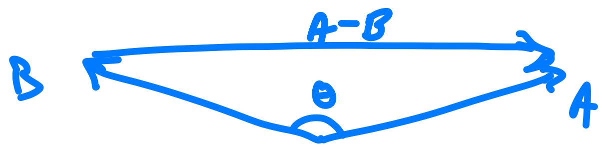
$$\|A-B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\| \cos\theta$$



$$\text{LHS} = (A-B)\cdot(A-B) = \|A\|^2 + \|B\|^2 - 2\underbrace{A\cdot B}_{=\|A\|\|B\|\cos\theta}$$

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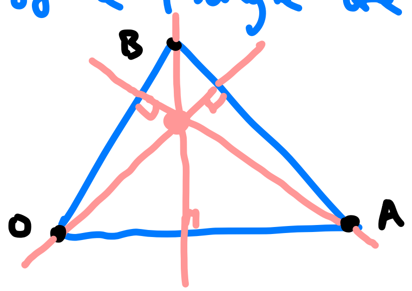

The 3 lines have equations (for P = pt. on line)

$$\left.\begin{aligned} P\cdot(B-A) &= 0 \\ (P-A)\cdot B &= 0 \\ (P-B)\cdot A &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} P\cdot B &= P\cdot A \\ P\cdot B &= A\cdot B \\ P\cdot A &= A\cdot B \end{aligned}\right.$$

which reduce to 2 equations. So a common solution exists.

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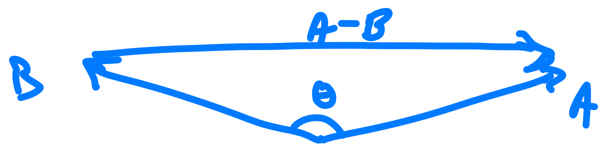

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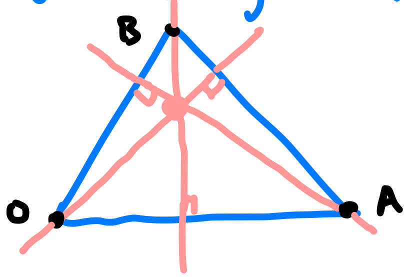

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§4. Subspaces

If $A_1, \dots, A_k \in \mathbb{R}^n$, a linear combination of them is $\sum_{i=1}^{k} c_i A_i$ $(c_i \in \mathbb{R})$.

Their span (or linear span) is the set of all these linear combinations. (If $\text{Span}(A_1, \dots, A_k) = \mathbb{R}^n$, then $A_1, \dots, A_k$ are said "to span $\mathbb{R}^n$".)

The span of a set of vectors is the basic example of a subspace.

Definition: A subset $V \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

- $0 \in V$,
- $\vec{v} \in V$ & $c \in \mathbb{R} \Rightarrow c\vec{v} \in V$, and
- $\vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V$

all hold.

§4. Subspaces

If $A_1, \ldots, A_k \in \mathbb{R}^n$, a linear combination of them is $\sum_{i=1}^{k} c_i A_i$ $(c_i \in \mathbb{R})$.

Their span (or linear span) is the set of all these linear combinations. (If $\mathrm{Span}(A_1, \ldots, A_k) = \mathbb{R}^n$, then $A_1, \ldots, A_k$ are said "to span $\mathbb{R}^n$.")

Example: let $\hat{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ ← ith place be the "ith standard basis vector" in $\mathbb{R}^n$.

Then $\hat{e}_1, \ldots, \hat{e}_n$ span $\mathbb{R}^n$: any $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ can be written as a linear combination of them

$$A = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \sum a_i \hat{e}_i$$

in exactly one way. $\square$

The span of a set of vectors is the basic example of a subspace.

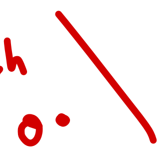
Definition: A subset $V \subseteq \mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

- $0 \in V$,
- $\vec{v} \in V$ & $c \in \mathbb{R} \Rightarrow c\vec{v} \in V$, and
- $\vec{v}, \vec{w} \in V \Rightarrow \vec{v} + \vec{w} \in V$

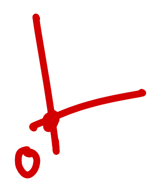
all hold.

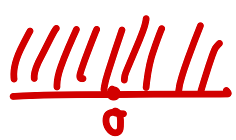
Non-Examples:

- the empty set $\emptyset$
- a line / plane / etc. not through the origin



- the union of two lines through the origin



- circles of any kind
- the upper half-plane  in $\mathbb{R}^2$

Examples:

- $\{0\}$ (the "trivial" subspace)
- $\mathbb{R}^n$ itself
- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$ $\quad 0 = \sum_i 0\vec{v}_i$

$$A = \sum_i a_i \vec{v}_i, \quad B = \sum_i b_i \vec{v}_i \Rightarrow A + B = \sum_i (a_i + b_i)\vec{v}_i$$

$$c \in \mathbb{R} \Rightarrow cA = \sum_i c a_i \vec{v}_i$$

if $k = 1$: line thru origin

if $k = 2$ and $\vec{v}_1, \vec{v}_2$ not parallel : plane thru origin

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if $k=2$ and $\vec{v}_1, \vec{v}_2$ not parallel :

plane thru origin

- $W := \{\vec{w} \in \mathbb{R}^n \mid \vec{w}\cdot\vec{v} = 0\}$, where $\underline{\vec{v} \in \mathbb{R}^n \setminus \{0\}}$

$0 \in W$; and given $A, B \in W$, $c \in \mathbb{R}$,

$(A+B)\cdot\vec{v} = A\cdot\vec{v} + B\cdot\vec{v} = 0+0 = 0$ and

$(cA)\cdot\vec{v} = c(A\cdot\vec{v}) = c0 = 0$

- More generally, given $V \subseteq \mathbb{R}^n$ subspace, the orthogonal complement

$$V^\perp := \{\vec{w} \in \mathbb{R}^n \mid \vec{w}\cdot\vec{v} = 0 \;\; \forall \vec{v} \in V\}$$

is a subspace of $\mathbb{R}^n$.

- $V = \left\{ \begin{pmatrix}1\\1\\-1\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$

At first, you say "hold on, this doesn't pass through 0" — until you realize that $\begin{pmatrix}1\\1\\-1\end{pmatrix} = -\begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}2\\0\\1\end{pmatrix}$,

so we can rewrite V as

$$\left\{ (s-1)\begin{pmatrix}1\\-1\\2\end{pmatrix} + (t+1)\begin{pmatrix}2\\0\\1\end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$
$$= \left\{ u\begin{pmatrix}1\\-1\\2\end{pmatrix} + v\begin{pmatrix}2\\0\\1\end{pmatrix} \;\middle|\; u,v \in \mathbb{R} \right\}$$
$$= \operatorname{Span}\left\{ \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\}.$$

Examples:

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- More generally, given $V \le \mathbb{R}^n$ subspace,

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For a couple more examples, we make the following

<u>Definition</u>: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a <u>linear transformation</u> if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad \forall \vec{u}, \vec{v} \in \mathbb{R}^n$

and

- $T(c\vec{u}) = cT(\vec{u}) \quad \forall c \in \mathbb{R}\ \&\ u \in \mathbb{R}^n$.

- $V = \left\{ \begin{pmatrix}1\\1\\-7\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$

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Then

$$\ker(T) := \{\vec{u} \in \mathbb{R}^n \mid T(\vec{u}) = \vec{0}\} \subset \mathbb{R}^n$$

and

$$\mathrm{im}(T) := \{T(\vec{u}) \mid \vec{u} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

are subspaces: e.g. if $\vec{u},\vec{v} \in \ker(T)$

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Examples of linear transformations:

- $T: \mathbb{R}^n \to \mathbb{R}$ given by $\vec{x} \mapsto \vec{v} \cdot \vec{x}$ (for some $\vec{v} \in \mathbb{R}^n$)

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... instead, we will show that all linear transformations can be represented by matrices, and that conversely all matrices produce linear transformations.

Then

$$\ker(T) := \{\vec{u}\in\mathbb{R}^n \mid T(\vec{u})=\vec{0}\} \subset \mathbb{R}^n$$

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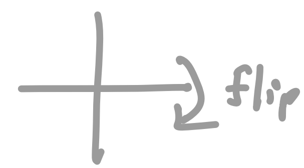

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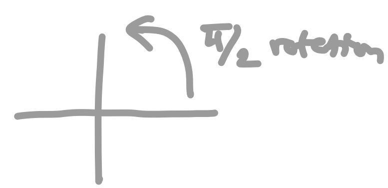

but it gets unwieldy to check by hand that such things satisfy the definition.

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Definition: The matrix of T is the matrix with columns

$$T(\hat{e}_1), T(\hat{e}_2), \ldots, T(\hat{e}_n).$$

Since these are in $\mathbb{R}^m$, this matrix $[T]$ has size $m \times n$ $\left(m \updownarrow \overset{\;n}{\longleftrightarrow} \right)$.

We will pick up from here next time

>———— • ————<

Then

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- $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$ $\quad [T] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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