Rank + Nullity
We have been discussing so far only $\mathbb{R}^n$ and its vector subspaces, and will for the most part be continuing along that path.

Some of what I'll discuss today, however, profits from being explored in more general terms: a (real) vector space is a set $V$ with a special element $\mathbf{0}$, and 3 operations:

- **Addition**: $+: V \times V \rightarrow V$
  \[(\mathbf{v}, \mathbf{w}) \rightarrow \mathbf{v} + \mathbf{w}\]

- **Scalar Multiplication**: $\cdot : \mathbb{R} \times V \rightarrow V$
  \[(r, \mathbf{v}) \rightarrow r\mathbf{v}\]

- **(Additive) Inversion**: $- : V \rightarrow V$
  \[\mathbf{v} \rightarrow -\mathbf{v}\]

Satisfying:

- “+” is commutative & associative
- $\mathbf{0} + \mathbf{v} = \mathbf{v} \text{ and } \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- $1 \mathbf{v} = \mathbf{v} \text{ and } r(s\mathbf{v}) = (rs)\mathbf{v}$
- $r(u\mathbf{v}) = ru\mathbf{v} + rv\mathbf{v}$ and $(r+s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$.

The elements of $V$ are called vectors. A vector subspace $U \subset V$ is a subset closed under $+, \cdot$, and $-.$

**Examples**

- $\mathbb{R}^n$ (of course)
- $\mathbb{M}_{m \times n}$: $m \times n$ matrices ($+ = $ matrix addition, $\cdot = $ scalar – matrix multi. plays no role)
- $\mathbb{P}_n$: polynomials of degree $\leq n$ ($\mathbb{P}_{n-1}$ is a subspace)
- $C^0(\mathbb{R})$: continuous functions on $\mathbb{R}$ ($C^1(\mathbb{R})$, $C^2(\mathbb{R})$ etc. subspaces)
What about linear transformations? In this context, these are just maps

\[ T : V \rightarrow W \]

between 2 vector spaces satisfying

\[ T(\vec{v}_1 + \vec{v}_2) = T\vec{v}_1 + T\vec{v}_2 \quad \text{and} \quad T(r\vec{v}) = rT\vec{v}. \]

**Examples:**

- \( V = \mathbb{R}^n, \ W = \mathbb{R}^m \)
  
  \( T \vec{v} := A\vec{v}, \ A = m \times n \ \text{matrix} \)

- \( V = W = \mathbb{R}^3 \)
  
  \( T = \frac{d}{dt} \) is linear, since
  
  \[ \frac{d}{dt}(aP + bQ) = a\frac{dP}{dt} + b\frac{dQ}{dt} \]

- \( V = W = C^0(\mathbb{R}) \)

  \( f \mapsto (Tf)(x) := \int_0^x f(t) \, dt \).

**Example 1. Two kinds of subspaces**

Let \( T : V \rightarrow W \) be a linear map.

- The image (or "range") of \( T \) is the set of all vectors in \( W \) that are "hit" by \( T \):

  \[ \text{im}(T) := \{ \vec{w} \in W \mid \exists \vec{v} \in V \ \text{such that} \ T\vec{v} = \vec{w} \} \subset W \]

- The kernel (or "nullspace") of \( T \) is the set of vectors in \( V \) "hit" by \( T \):

  \[ \ker(T) := \{ \vec{v} \in V \mid T\vec{v} = \vec{0} \} \subset V \]

**Examples:**

- \( V = \mathbb{R}^n \rightarrow \mathbb{R}^m \)
  
  \( x \mapsto A\vec{x} \)

  \[ \ker(T) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} =: \text{nullspace of } A \]

- \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \)

  \[ x \mapsto A\vec{x} \]

  \[ \ker(T) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} =: \text{nullspace of } A \]
\[ \text{im}(T) = \{ b \in \mathbb{R}^n \mid b = A\xi \text{ for some } \xi \in \mathbb{R}^n \} = \overline{\text{Col}(A)} \subseteq \mathbb{R}^n \text{ column span of } A \]

- \( \frac{d}{dt} : \mathbb{P}_3 \rightarrow \mathbb{P}_3 \)
  \[ \ker\left(\frac{d}{dt}\right) = \{ \mathbf{p} \in \mathbb{P}_3 \mid \frac{d\mathbf{p}}{dt} = 0 \} = \mathbb{P}_0 = (1) \]
  consists of the constant polynomials;
  \[ \text{im}\left(\frac{d}{dt}\right) = \{ Q \in \mathbb{P}_3 \mid Q = \frac{d\mathbf{p}}{dt} \text{ for some } \mathbf{p} \in \mathbb{P}_3 \} = \mathbb{P}_2 \]

- \( \int_0^x = T : \mathbb{C}^0(\mathbb{R}) \rightarrow \mathbb{C}^0(\mathbb{R}) \)

\[ \ker(T) = \{ f \in \mathbb{C}^0(\mathbb{R}) \mid \int_0^x f(t) \, dt \text{ is identically zero} \} = \{ 0 \} \]

\[ \text{im}(T) = \{ g \in \mathbb{C}^0(\mathbb{R}) \mid g(x) = \int_0^x f(t) \, dt \text{ for some } f \in \mathbb{C}^0(\mathbb{R}) \} = \mathbb{C}^1(\mathbb{R}) \text{ by the FTC} \]

Let \( T : V \rightarrow W \) be a linear map.

**Proposition A:** \( \text{im}(T) \subseteq W \) & \( \ker(T) \subseteq V \) are subspaces.

**Proof:** If \( w, w' \in \text{im}(T) \), then \( \{ w = T v, w' = T v' \} \) for some \( v, v' \in V \). By linearity,

\[ T(av + bv') = aTv + bTv' = aw + bw' \Rightarrow aw + bw' \in \text{im}(T). \]

If \( v, v' \in \ker(T) \), then \( \{ Tv = 0, Tv' = 0 \} \)

So \( T(av + bv') = aTv + bTv' = 0 \)

\( \Rightarrow av + bv' \in \ker(T). \) \( \square \)

**Definition:** \( T \) is onto if \( \text{im}(T) = W. \)

\( T \) is 1-to-1 if \( v \neq v' \Rightarrow Tv \neq Tv' \) (i.e. \( Tv = Tv' \Rightarrow v = v' \)).

**Proposition B:** \( T \) is 1-1 \( \iff \ker(T) = \{ 0 \}. \)
\textbf{Proof:} $(\Rightarrow)$ is clear: only $\vec{0}$ can go to $\vec{0}$.

$(\Leftarrow)$ Suppose $\ker(T) = \{\vec{0}\}$, and let $Tv = Tv'$. Then by linearity,

$0 = Tv - Tv' = T(v-v') \Rightarrow v - v' \in \ker(T) = \{\vec{0}\} \Rightarrow v - v' = 0 \Rightarrow v = v'$.

\textbf{Examples:}

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is 1-1 $\iff$ Null($A$) = $\{\vec{0}\}$
  \hspace{3mm}$\vec{x} \mapsto Ax$ onto $\iff$ Col($A$) = $\mathbb{R}^m$

- $\frac{d}{dx} : P_3 \to P_3$ is neither 1-1 nor onto.

- $\int^x : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ is 1-1 but not onto.

How do we find \textit{Null} & \textit{Column spaces} of a matrix $A$?

\textbf{Let $T : V \to W$ be a linear map.}

\textbf{Proposition A:} im($T$) $\subseteq W$ & ker($T$) $\subseteq V$ are subspaces.

\textbf{Proof:} If $w, w' \in \text{im}(T)$, then \($\{w = Tv, w = Tv'\}$ for some $v, v' \in V$.

By linearity,

$T(av + bv') = aTv + bTv' = aw + bw' \Rightarrow av + bw' \in \text{im}(T)$.

If $v, v' \in \ker(T)$, then \($\{Tv = 0, Tv' = 0\}$.

So $T(av + bv') = aTv + bTv' = 0 \Rightarrow av + bv' \in \ker(T)$.

\textbf{Definition:} $T$ is \textit{onto} if $\text{im}(T) = W$.

$T$ is \textit{1-to-1} if $v \neq v' \Rightarrow Tv \neq Tv'$ (i.e. $Tv = Tv' \Rightarrow v = v'$).

\textbf{Proposition B:} $T$ is 1-1 $\iff$ ker($T$) = $\{\vec{0}\}$.
Ex 1/ Let $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.
(a) Is $\hat{v} = \begin{pmatrix} 6 \\ 5 \\ 0 \\ 0 \end{pmatrix}$ in $\text{Nul}(A)$?
(b) Parametrically describe $\text{Nul}(A)$.

Ex 2/ Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 1 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$.
(a) Is $\hat{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\text{Col}(A)$?
(b) Parametrically describe $\text{Col}(A)$. 
Ex 1 / Let \( A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \).

(a) Is \( \mathbf{v} = \begin{pmatrix} 6 \\ 5 \\ 0 \end{pmatrix} \) in \( \text{Null}(A) \)?

(b) Parametrically describe \( \text{Null}(A) \).

(a) Yes. Just compute \( A\mathbf{v} = \mathbf{0} \).

(b) We want all the solutions to \( A\mathbf{x} = \mathbf{0} \), and we know how to describe them parametrically since \( A \) is already (almost) in RREF: \( x_2, x_4 \) are non-pivot variables, thus free; we have:
\[
\begin{align*}
2x_1 - 4x_2 - 2x_3 + 0x_4 &= 0 \\
x_3 &= 5x_4 \\
x_5 &= 0
\end{align*}
\]

So, yes.

\[
\text{Null}(A) = \left\{ x_2 \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \big| x_2, x_4 \in \mathbb{R} \right\}
\]

In contrast to \( \text{Null}(A) \), it's easy to construct vectors in \( \text{Col}(A) \), but requires work to check that a given vector is in it.
2. Rank + Nullity

Let $A$ be an $m \times n$ matrix.

In particular, $A$ has $n$ columns.

Recall that the pivot columns of $A$ furnish a basis of the column space
$\text{Col}(A) \subseteq \mathbb{R}^m$;
while the the null space
$\text{Nul}(A) \subseteq \mathbb{R}^n$ comprising solutions of $Ax = \mathbf{0}$ has
a basis whose elements correspond to
the non-pivot variables (since they
parametrize the general solution).

The first part of the following
definition should by now be familiar:

**Definition:**
1. The rank of $A$ is
   \[
   \text{rank}(A) := \dim(\text{Col}(A)) = \# \text{ of pivot columns} = \# \text{ of leading 1's in } \text{rref}(A).
   \]
2. The nullity of $A$ is
   \[
   \text{nullity}(A) := \dim(\text{Nul}(A)) = \# \text{ of free variables} = \# \text{ of non-pivot columns}.
   \]

**Theorem:**
\[\text{rank}(A) + \text{nullity}(A) = n.\]

**Proof:**
\[
\# \text{ of pivot columns} + \# \text{ of non-pivot columns} = \# \text{ of columns}.
\]

$T : V \to W$ \[\dim(\text{im}T) + \dim(\text{ker}T) = \dim(V) \]
Ex 3/ Suppose that a system of 8 homogeneous linear equations in 15 unknowns
\[
\begin{cases}
    a_1x_1 + \cdots + a_7x_7 + a_8x_8 = 0 \\
    \vdots \\
    a_9x_9 + \cdots + a_{15}x_{15} = 0
\end{cases}
\]
has (exactly) 10 independent solutions. What then is the dimension of the space of vectors \( \mathbf{b} \in \mathbb{R}^9 \) for which
\[
\begin{cases}
    a_1x_1 + \cdots + a_7x_7 + a_8x_8 = b_1 \\
    \vdots \\
    a_9x_9 + \cdots + a_{15}x_{15} = b_9
\end{cases}
\]
is consistent?

Definition: (a) The rank of \( A \) is
\[
\text{rank} \ (A) := \dim (\text{Col}(A)) = \# \text{ of pivot columns} = \# \text{ of leading 1's in } r(A).
\]
(b) The nullity of \( A \) is
\[
\text{nullity} \ (A) := \dim (\text{Null}(A)) = \# \text{ of free variables} = \# \text{ of non-pivot columns}.
\]

Theorem: rank \( (A) + \text{nullity} \ (A) = n \).

Proof: \# of pivot columns
\[
+ \# \text{ of non-pivot columns}
\]
\[
= \# \text{ of columns}.
\]
Ex 3/ Suppose that a system of 8 homogeneous linear equations in 15 unknowns
\[
\begin{align*}
\sum_{i=1}^{8} a_i x_i &= 0 \\
\vdots \\
\sum_{i=1}^{8} a_{8,15} x_{15} &= 0
\end{align*}
\]
has (exactly) 10 independent solutions. What then is the dimension of the space of vectors \( \mathbf{b} \in \mathbb{R}^8 \) for which
\[
\begin{align*}
\sum_{i=1}^{8} a_i x_i &= b_1 \\
\vdots \\
\sum_{i=1}^{8} a_{8,15} x_{15} &= b_8
\end{align*}
\]
is consistent?

Here \( \mathbf{b} \) must lie in \( \text{Col}(A) \), so the answer is
\[
\dim(\text{Col}(A)) = \text{rank}(A) = 15 - \text{nullity}(A) = 15 - 10 = 5.
\]

Q: What is the largest possible rank of an \( m \times n \) matrix?

Well, \( \text{rank}(A) = \# \) of pivot columns
\[
= \# \text{ of leading } 1's \text{ in } \text{ref}(A)
\]
\[
= \# \text{ of nonzero rows in } \text{ref}(A)
\]
\[
\Rightarrow \text{rank}(A) \leq \min \{m,n\}.
\]
When equal, we say that \( A \) has maximal rank.

Q: What does maximal rank "look like"?

There are 3 possible cases:
\[
\begin{align*}
\text{(m \geq n)} & \quad \begin{pmatrix} A \end{pmatrix} \\
\text{If } \text{rank}(A) = n, \text{ then} \\
\text{nullity}(A) = 0, \text{ and} \\
\hat{x} \rightarrow A\hat{x} \text{ is 1-to-1.}
\end{align*}
\]
\[
\begin{align*}
\text{(m < n)} & \quad \begin{pmatrix} A \end{pmatrix} \\
\text{If } \text{rank}(A) = m, \text{ then} \\
m = \text{dim} (\text{Col}(A)) \\
\text{so } \hat{x} \rightarrow A\hat{x} \text{ is onto.}
\end{align*}
\]
\[
\begin{align*}
\text{(m = n)} & \quad \begin{pmatrix} A \end{pmatrix} \\
\text{If } \text{rank}(A) = n, \text{ then} \\
\text{ref}(A) = I_n, \text{ A is} \\
\text{invertible, and } \hat{x} \rightarrow A\hat{x} \\
\text{is 1-to-1 and onto.}
\end{align*}
\]
3. Row spaces

Let \( A \) be an \( m \times n \) matrix.

**Definition:** \( \text{Row}(A) \subseteq \mathbb{R}^m \) is the subspace consisting of all linear combinations of rows of \( A \). (One can also think of this as \( \text{C}(\mathbf{A})^\top \).)

Like \( \text{Nul}(A) \), and unlike \( \text{Col}(A) \), \( \text{Row}(A) \) is unchanged by EROs; so \( \text{Row}(\text{rref}(A)) = \text{Row}(A) \).

**Ex 4** \[
A = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \end{pmatrix}
\]

Find bases of dimensions of \( \text{Col}(A) \), \( \text{Nul}(A) \), \( \text{Row}(A) = \text{Col}(A^\top) \), and \( \text{Nul}(A^\top) \).

We could flip \( A \) to \( A^\top \) and compute \( \text{Col}(A^\top) \) & \( \text{Nul}(A^\top) \) directly, by row-reducing \( A^\top \). But there is a way to do an 4 just by row-reducing \( A \):

\[
A \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{rref}(A)
\]

... so a basis for \( \text{Col}(A) \) is given by the 1st 2 columns of \( A \), i.e. \( \{(1), (2)\} \Rightarrow \text{nullity}(A) = 2 = \dim(\text{Col}(A)) \).

... and a basis for \( \text{Nul}(A) = \{ \text{solutions to } \begin{cases} x_1 + x_2 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \text{ with free } x_1 \text{ and } x_4 \} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \text{nullity}(A) = 3 = \dim(\text{Nul}(A)) \).

... and a basis for \( \text{Row}(A) \) by the nonzero rows \( \{(1,0,1,1,0), (0,1,1,0,0)\} \) of \( \text{rref}(A) \).

... while for \( \text{Nul}(A^\top) \), the basis is just \( \{(1)\} \) since \( -R_1 + R_3 = 0 \).
Notice in this example that $2+3=5$ is rank + nullity for $A$, while $2+1=\dim(\text{Row}(A)) + \dim(\text{Null}(A)) = 3$ is rank + nullity for $^tA$.

- In general, you can find a basis for $\text{Null}(^tA)$ by using augmented matrix:

$$
\begin{bmatrix}
A & I_3
\end{bmatrix} \xrightarrow{\text{row- ops}} \begin{bmatrix}
1 & 0 & 1 & 0 & -1 & 2 & 0 \\
0 & 1 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

The rows in the $3\times3$ to the right of the bar following rows of all 0s in $\text{ref}(A)$, yield a basis for $\text{Null}(^tA)$.

- $\dim(\text{Row}(A)) = \text{rank}(A) = \dim(\text{Col}(A))$.

b/c the nonzero rows of $\text{ref}(A)$ are a basis of Row$(A)$ (they're independent, and Span Row$(A)$ by removing the ER0s), and there's one for every leading '1'.

Ex 5/ Consider a homogeneous system of 12 linear equations in 8 unknowns with 2 independent solutions (4 no more). Can we reduce the # of equations?
where the \( \cong \) symbol means that the restriction of the transformation to \( \text{Row}(A) \to \text{Col}(A) \) is an isomorphism, i.e. a 1-to-1 and onto map.

To see this, note that

\[
\text{Row}(A) = \text{Nul}(A)^\perp
\]

since \( A\vec{v} = \vec{0} \iff \vec{r} \cdot \vec{v} = 0 \) for each \( \vec{r} \).

(As \( \text{Col}(A) = \text{Row}(A) \) and \( \text{Col}(A) = \text{Nul}(A)^\perp \).)

So if \( \vec{v} \in \text{Row}(A) \) has \( A\vec{v} = \vec{0} \), it is in \( \text{Nul}(A) \) and is orthogonal to itself: \( \vec{v} \cdot \vec{v} = 0 \). But then \( \vec{v} = \vec{0} \). Hence \((*) \) is 1-to-1; and since \( \text{Row}(A) \) & \( \text{Col}(A) \) both have dimension \( r \), it is also onto (why?).

This picture is closely related to least squares/linear regression and the singular value decomposition.

Ex 5/ Consider a homogeneous system of \( 12 \) linear equations in \( 8 \) unknowns with 2 independent solutions (and no more). Can we reduce the \# of equations?

Represent as \( A\vec{x} = \vec{0} \), \( A \) \( 12 \times 8 \).

A basis for \( \text{Row}(A) \) corresponds to a “minimal” set of equations.

The number thereof = \( \text{rank}(A) = 8 - \text{nullity}(A) = 8 - 2 = 6 \). So indeed, we can eliminate half of the original equations. //

To summarize our discussion of the 4 subspaces above, we have the picture:

\[
\begin{array}{c}
\mathbb{R}^n \\
\xrightarrow{A \cong} \\
\mathbb{R}^m
\end{array}
\]