RANK + NULLITY

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We have been discussing so far only $\mathbb{R}^n$ and its vector subspaces, and will for the most part be continuing along that path.

Some of what I'll discuss today, however, profits from being explored in more general terms: a (real) vector space is a set V with a special element $\vec{0}$, and 3 operations:

- ADDITION: $+: V \times V \to V$
  $(\vec{v}, \vec{w}) \longmapsto \vec{v} + \vec{w}$
- SCALAR MULTIPLICATION: $\cdot: \mathbb{R} \times V \to V$
  $(r, \vec{v}) \longmapsto r\vec{v}$
- (ADDITIVE) INVERSION: $-: V \to V$
  $\vec{v} \longmapsto -\vec{v}$

satisfying

- "$+$" is commutative & associative
- $\vec{0} + \vec{v} = \vec{v}$ and $\vec{v} + (-\vec{v}) = \vec{0}$
- $1\vec{v} = \vec{v}$ and $r(s\vec{v}) = (rs)\vec{v}$
- $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$ and $(r+s)\vec{v} = r\vec{v} + s\vec{v}$.

The elements of V are called vectors.

A vector subspace $U \subset V$ is a subset closed under $+$, $\cdot$, and $-$.

EXAMPLES

- $\mathbb{R}^n$ (of course)
- $M_{m \times n}$ = $m \times n$ matrices ($+$ = matrix addition, $\cdot$ = scalar mult. — matrix mult. plays no role)
- $\mathbb{P}_n$ = polynomials of degree $\leq n$
  ($\mathbb{P}_{n-1}$ is a subspace)
- $C^0(\mathbb{R})$ = continuous functions on $\mathbb{R}$
  ($C^1(\mathbb{R})$, $C^2(\mathbb{R})$ etc. subspaces)

What about linear transformations?

In this context, these are just maps

$$T : V \to W$$

between 2 vector spaces satisfying

$T(\vec{v}_1+\vec{v}_2) = T\vec{v}_1 + T\vec{v}_2$ and $T(r\vec{v}) = rT\vec{v}$.

EXAMPLES:

- $V = \mathbb{R}^n$, $W = \mathbb{R}^m$
  $T\vec{v} := A\vec{v}$, $A = m\times n$ matrix
- $V = W = \mathbb{P}_3$
  $T = \frac{d}{dt}$ is linear, since
  $$\frac{d}{dt}(aP+bQ) = a\frac{dP}{dt} + b\frac{dQ}{dt}$$
- $V = W = C^0(\mathbb{R})$
  $$f \mapsto (Tf)(x) := \int_0^x f(t)\,dt.$$

§1. Two kinds of subspaces

Let $T : V \to W$ be a linear map.

- The image (or "range") of T is the set of all vectors in W that are "hit" by T:
  $$\mathrm{im}(T) := \{\vec{w} \in W \mid \vec{w} = T\vec{v} \text{ for some } \vec{v} \in V\} \subset W$$
- The kernel (or "nullspace") of T is the set of vectors in V "killed" by T:
  $$\ker(T) := \{\vec{v} \in V \mid T\vec{v} = \vec{0}\} \subset V$$

EXAMPLES:

- $T : \mathbb{R}^n \to \mathbb{R}^m$
  $\vec{x} \mapsto A\vec{x}$

or $N(A)$

$$\ker(T) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} =: \mathrm{Nul}(A) \subseteq \mathbb{R}^n$$

nullspace of A

$$\operatorname{im}(T) = \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$$
$$=: \underline{\operatorname{Col}(A)} \subseteq \mathbb{R}^m$$

column space of A

= span of A's columns

or $C(A)$

- $\frac{d}{dt} : \mathbb{P}_3 \to \mathbb{P}_3$

$$\ker\left(\frac{d}{dt}\right) = \left\{P \in \mathbb{P}_3 \;\middle|\; \frac{dP}{dt} = 0\right\} = \mathbb{P}_0 \; (= \mathbb{R})$$

consists of the constant polynomials

$$\operatorname{im}\left(\frac{d}{dt}\right) = \left\{Q \in \mathbb{P}_3 \;\middle|\; Q = \frac{dP}{dt} \text{ for some } P \in \mathbb{P}_3\right\}$$
$$= \mathbb{P}_2$$

- "$\int_0^x$" $= T : C^0(\mathbb{R}) \to C^0(\mathbb{R})$

$$\ker(T) = \left\{f \in C^0(\mathbb{R}) \;\middle|\; \int_0^x f(t)\,dt \text{ is identically zero}\right\}$$
$$= \{0\}$$

$$\operatorname{im}(T) = \left\{g \in C^0(\mathbb{R}) \;\middle|\; g(x) = \int_0^x f(t)\,dt \text{ for some } f \in C^0(\mathbb{R})\right\}$$
$$= C^1(\mathbb{R}) \text{ by the FTC}$$

continuously differentiable fcns.

Let $T : V \to W$ be a linear map.

$$\boxed{\text{\underline{Proposition A}: } \operatorname{im}(T) \subseteq W \text{ \& } \ker(T) \subseteq V \text{ are subspaces.}}$$

__Proof__: If $w, w' \in \operatorname{im}(T)$, then $\begin{cases} w = Tv \\ w' = Tv' \end{cases}$ for some $v, v' \in V$. By linearity,

$$T(av + bv') = aTv + bTv' = aw + bw'$$
$$\Rightarrow aw + bw' \in \operatorname{im}(T).$$

If $v, v' \in \ker(T)$, then $\begin{cases} Tv = 0 \\ Tv' = 0. \end{cases}$

So $T(av + bv') = aTv + bTv' = 0$

$\Rightarrow av + bv' \in \ker(T)$. $\square$

$$\boxed{\begin{array}{l} \text{\underline{Definition}: } T \text{ is onto if } \operatorname{im}(T) = W. \\ T \text{ is 1-to-1 if } v \neq v' \Rightarrow Tv \neq Tv' \\ \quad (\text{i.e. } Tv = Tv' \Rightarrow v = v'). \end{array}}$$

$$\boxed{\text{\underline{Proposition B}: } T \text{ is 1-1} \Leftrightarrow \ker(T) = \{0\}.}$$

Proof: ($\Rightarrow$) is clear: only $\vec{0}$ can go to $\vec{0}$.

($\Leftarrow$) Suppose $\ker(T) = \{0\}$, and let $Tv = Tv'$. Then by linearity,

$$0 = Tv - Tv' = T(v - v') \Rightarrow v - v' \in \underbrace{\ker(T)}_{=\{0\}}$$

$\Rightarrow v - v' = 0 \Rightarrow v = v'$. $\square$

Examples:

- $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$ is 1-1 $\Leftrightarrow \mathrm{Nul}(A) = \{0\}$
  onto $\Leftrightarrow \mathrm{Col}(A) = \mathbb{R}^m$
- $\frac{d}{dt}: \mathbb{P}_3 \to \mathbb{P}_3$ is neither 1-1 nor onto
- $\int_0^x : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ is 1-1 but not onto

How do we find null & column spaces of a matrix A?

Let $T: V \to W$ be a linear map.

Proposition A: $\mathrm{im}(T) \subseteq W$ & $\ker(T) \subseteq V$ are subspaces.

Proof: If $w, w' \in \mathrm{im}(T)$, then $\begin{cases} w = Tv \\ w' = Tv' \end{cases}$ for some $v, v' \in V$. By linearity,

$$T(av + bv') = aTv + bTv' = aw + bw'$$

$\Rightarrow aw + bw' \in \mathrm{im}(T)$.

If $v, v' \in \ker(T)$, then $\begin{cases} Tv = 0 \\ Tv' = 0. \end{cases}$

So $T(av + bv') = aTv + bTv' = 0$

$\Rightarrow av + bv' \in \ker(T)$. $\square$

Definition: T is onto if $\mathrm{im}(T) = W$.

T is 1-to-1 if $v \neq v' \Rightarrow Tv \neq Tv'$
(i.e. $Tv = Tv' \Rightarrow v = v'$).

Proposition B: T is 1-1 $\Leftrightarrow \ker(T) = \{0\}$.

Ex 1/ Let $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

(a) Is $\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \\ 1 \\ 0 \end{pmatrix}$ in Nul(A)?

(b) Parametrically describe Nul(A).

Ex 2/ Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 1 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$.

(a) Is $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ in Col(A)?

(b) Parametrically describe Col(A).

Ex 1/ Let $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

(a) Is $\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \\ 1 \\ 0 \end{pmatrix}$ in $\mathrm{Nul}(A)$?

(b) Parametrically describe $\mathrm{Nul}(A)$.

(a) Yes. Just compute $A\vec{v} = \vec{0}$.

(b) We want all the solutions to $A\vec{x} = \vec{0}$, and we know how to describe them parametrically since A is already (almost) in RREF: x_2, x_4 are non-pivot-variables & thus free; we have

$$\begin{cases} x_1 = 4x_2 - 2x_4 \\ x_3 = 5x_4 \\ x_5 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4x_2 - 2x_4 \\ x_2 \\ 5x_4 \\ x_4 \\ 0 \end{pmatrix} \Rightarrow$$

$$\mathrm{Nul}(A) = \left\{ x_2 \begin{pmatrix} 4 \\ \boxed{1} \\ 0 \\ \boxed{0} \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ \boxed{0} \\ 5 \\ \boxed{1} \\ 0 \end{pmatrix} \,\middle|\, x_2, x_4 \in \mathbb{R} \right\}$$

basis of Nul(A) $\leftrightarrow$ free variables

Ex 2/ Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 1 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$.

(a) Is $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ in $\mathrm{Col}(A)$?

(b) Parametrically describe $\mathrm{Col}(A)$.

(a) Need to see if $A\vec{x} = \vec{v}$ is consistent:

$$\left[\begin{array}{cc|c} 6 & -4 & 0 \\ -3 & 1 & 1 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 1 & 1 \\ -3 & 2 & 0 \\ -3 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

which yields $\begin{cases} -x_2 = 1 \\ 3x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1 \end{pmatrix}$.

So, <u>yes</u>.

(b) Easy: $\mathrm{Col}(A) = \left\{ s\begin{pmatrix} 6 \\ -3 \\ -9 \\ 9 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \\ 6 \\ -6 \end{pmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$

In contrast to $\mathrm{Nul}(A)$, it's <u>easy</u> to <u>construct</u> vectors in $\mathrm{Col}(A)$, but requires work to check that a given vector is in it.

§ 2. Rank + Nullity

Let A be an $m \times n$ matrix.

In particular, A has n columns.

Recall that the pivot columns of A furnish a basis of the column space

$$\mathrm{Col}(A) \subseteq \mathbb{R}^m ;$$

while the the null space

$$\mathrm{Nul}(A) \subseteq \mathbb{R}^n$$

comprising solutions of $A\vec{x} = \vec{0}$ has a basis whose elements correspond to the non-pivot variables (since they parametrize the general solution).

The first part of the following definition should by now be familiar:

Definition: (a) The rank of A is

$$\begin{aligned}\mathrm{rank}(A) &:= \dim(\mathrm{Col}(A)) \\ &= \#\text{ of pivot columns} \\ &= \#\text{ of leading 1's in rref}(A).\end{aligned}$$

(b) The nullity of A is

$$\begin{aligned}\mathrm{nullity}(A) &:= \dim(\mathrm{Nul}(A)) \\ &= \#\text{ of free variables} \\ &= \#\text{ of non-pivot columns}.\end{aligned}$$

Theorem: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

Proof:
$$\begin{array}{l} \#\text{ of pivot columns} \\ + \ \#\text{ of non-pivot columns} \\ \hline = \#\text{ of columns}. \end{array}$$
$\square$

$T: V \to W \qquad \dim[\mathrm{im}\, T] + \dim(\ker T) = \dim(V)$

Ex 3/ Suppose that a system of 8 homogeneous linear equations in 15 unknowns

$$\begin{cases} a_{11}x_1 + \cdots + a_{1,15}x_{15} = 0 \\ \vdots \\ a_{81}x_1 + \cdots + a_{8,15}x_{15} = 0 \end{cases}$$

has (exactly) 10 independent solutions. What then is the dimension of the space of vectors $\vec{b} \in \mathbb{R}^8$ for which

$$\begin{cases} a_{11}x_1 + \cdots + a_{1,15}x_{15} = b_1 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{81}x_1 + \cdots + a_{8,15}x_{15} = b_8 \end{cases}$$

is consistent?

Definition: (a) The rank of A is

$$\begin{aligned} \operatorname{rank}(A) &:= \dim(\operatorname{Col}(A)) \\ &= \#\text{ of pivot columns} \\ &= \#\text{ of leading 1's in } \operatorname{rref}(A). \end{aligned}$$

(b) The nullity of A is

$$\begin{aligned} \operatorname{nullity}(A) &:= \dim(\operatorname{Nul}(A)) \\ &= \#\text{ of free variables} \\ &= \#\text{ of non-pivot columns}. \end{aligned}$$

Theorem: $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

Proof: # of pivot columns

+ # of non-pivot columns

= # of columns. $\square$

Ex 3/ Suppose that a system of 8 homogeneous linear equations in 15 unknowns

$$\begin{cases} a_{11}x_1 + \cdots + a_{1,15}x_{15} = 0 \\ \vdots \\ a_{81}x_1 + \cdots + a_{8,15}x_{15} = 0 \end{cases}$$

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is consistent?

Here $\vec{b}$ must lie in Col(A), so the answer is

$$\dim(\text{Col}(A)) = \text{rank}(A) = n - \text{nullity}(A)$$
$$= 15 - 10 = 5. \quad //$$

Q: What is the largest possible rank of an $m \times n$ matrix?

Well, $\text{rank}(A) = \#$ of pivot columns
$= \#$ of leading 1's in rref(A)
$= \#$ of nonzero rows in rref(A)

$\Rightarrow \text{rank}(A) \leq \min\{m, n\}$. When equal, we say that A has maximal rank.

Q: What does maximal rank "look like"?

There are 3 possible cases:

$(m > n)$ $\begin{pmatrix} \\ A \\ \\ \end{pmatrix}$ If $\text{rank}(A) = n$, then $\text{nullity}(A) = 0$, and $\vec{x} \mapsto A\vec{x}$ is 1-to-1.

$(m < n)$ $\begin{pmatrix} \quad A \quad \end{pmatrix}$ If $\text{rank}(A) = m$, then $m = \dim(\text{Col}(A))$, so $\vec{x} \mapsto A\vec{x}$ is onto.

$(m = n)$ $\begin{pmatrix} A \end{pmatrix}$ If $\text{rank}(A) = n$, then $\text{rref}(A) = I_n$, A is invertible, and $\vec{x} \mapsto A\vec{x}$ is 1-to-1 and onto.

§3. Row spaces

Let A be an $m \times n$ matrix.

Definition: $\text{Row}(A) \subset \mathbb{R}^n$ (or $R(A)$) is the subspace consisting of all linear combinations of rows of A. (One can also think of this as $C({}^tA)$.)

Like $\text{Nul}(A)$, and unlike $\text{Col}(A)$, $\text{Row}(A)$ is unchanged by EROs, so $\text{Row}(\text{rref}(A)) = \text{Row}(A)$.

Ex 4/ $A = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \end{pmatrix}$. Find bases & dimensions of $\text{Col}(A)$, $\text{Nul}(A)$, $\text{Row}(A)$ $(= \text{Col}({}^tA))$, and $\text{Nul}({}^tA)$.

dependencies on columns of A

dependencies on rows of A

We could flip $A \rightsquigarrow {}^tA$ and compute $\text{Col}({}^tA)$ & $\text{Nul}({}^tA)$ directly, by row-reducing ${}^tA$. But there is a way to do all 4 just by row-reducing A:

$$A \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} \boxed{1} & 0 & 1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \text{rref}(A)$$

... so a basis for $\text{Col}(A)$ is given by the 1^{st} + 2^{nd} columns of A, i.e. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$

$\Rightarrow \text{rank}(A) = 2 = \dim(\text{Col}(A))$.

... and a basis for $\text{Nul}(A) =$

$$\left(\text{solutions to } \begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \text{ with the free variables } \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} -1 \\ -1 \\ \boxed{1} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ \boxed{1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \boxed{1} \end{pmatrix} \right\}$$

$\Rightarrow \text{nullity}(A) = 3 = \dim(\text{Nul}(A))$.

... and a basis for $\text{Row}(A)$ by the nonzero rows $\{(1,0,1,1,0), (0,1,1,0,0)\}$ of $\text{rref}(A)$

... while for $\text{Nul}({}^tA)$, the basis is just $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ since $-\vec{R}_1 + \vec{R}_3 = \vec{0}$. //

- Notice in this example that $2+3=5$ is rank + nullity for A, while $2+1=$ $\dim(\text{Row}(A)) + \dim(\text{Nul}({}^tA)) = 3$ is rank + nullity for ${}^tA$.

- In general, you can find a basis for $\text{Nul}({}^tA)$ by using augmented matrices:

$$[A \mid I_3] \xrightarrow[\text{ops}]{\text{row-}} \left[\begin{array}{ccccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

these keep a record of row-operations!

The rows in the 3×3 to the right of the bar following rows of all 0's in rref(A), yield a basis for $\text{Nul}({}^tA)$.

$$\vec{R}_1(\text{rref}) = -\vec{R}_1 + 2\vec{R}_2$$
$$\vec{R}_2(\text{rref}) = \vec{R}_1 - \vec{R}_2$$
$$\underbrace{0 = \vec{R}_3(\text{rref}) = -\vec{R}_1 + \vec{R}_3}_{\text{dependency}}$$

(This is why ⟿ dependency)

- $\dim(\text{Row}(A)) = \text{rank}(A)$ $(= \dim(\text{Col}(A)))$

b/c the nonzero rows of rref(A) are a basis of Row(A) (they're independent, & span Row(A) by reversing the EROs), & there's one for every leading '1'.

Ex 5/ Consider a homogeneous system of 12 linear equations in 8 unknowns with 2 independent solutions (& no more). Can we reduce the # of equations?

where the "$\cong$" symbol means that the restriction of the transformation to $\boxed{Row(A) \xrightarrow{(*)} Col(A)}$ is an isomorphism, i.e. a 1-to-1 and onto map.

To see this, note that

$$Row(A) = Nul(A)^{\perp}$$

since $A\vec{v} = \vec{0} \iff \vec{R}_i \cdot \vec{v} = 0$ for each i.

(As $Col(A) = Row({}^tA)$, $Col(A) = Nul({}^tA)^{\perp}$.)

So if $\vec{v} \in Row(A)$ has $A\vec{v} = \vec{0}$, it is in $Nul(A)$ & is orthogonal to itself: $\vec{v} \cdot \vec{v} = 0$. But then $\vec{v} = \vec{0}$. Hence $(*)$ is 1-to-1; and since $Row(A)$ & $Col(A)$ both have dimension r, it is also onto (why?).

This picture is closely related to least squares/linear regression and the singular value decomposition.

Ex 5/ Consider a homogeneous system of 12 linear equations in 8 unknowns with 2 independent solutions (& no more). Can we reduce the # of equations?

Represent as $A\vec{x} = \vec{0}$, A 12×8.

A basis for $Row(A)$ corresponds to a "minimal" set of equations.

The number thereof $= rank(A) = 8 - nullity(A) = 8 - 2 = 6$. So indeed, we can eliminate half of the original equations. //

To summarize our discussion of the 4 subspaces above, we have the picture: