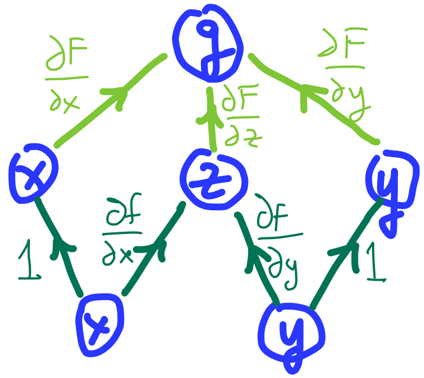
MANIFOLDS II

We have spent some time discussing linear subspaces of $\mathbb{R}^n$. Today we return to the world of Calculus with a first look at manifolds, or "nonlinear subspaces which are smooth at each point". The main examples here are level sets of C^1 functions and intersections of them. Implicit differentiation plays an important role, so we'll begin with that.

§1. Implicit partial differentiation

Using the "Chain rule diagram" shown,



we obtain
$$\begin{cases} 0 = \dfrac{\partial g}{\partial x} = \dfrac{\partial F}{\partial x}\cdot 1 + \dfrac{\partial F}{\partial z}\cdot\dfrac{\partial f}{\partial x} \\[2ex] 0 = \dfrac{\partial g}{\partial y} = \dfrac{\partial F}{\partial y}\cdot 1 + \dfrac{\partial F}{\partial z}\cdot\dfrac{\partial f}{\partial y} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{-\partial F/\partial x}{\partial F/\partial z} = \dfrac{-z}{x+2z-e^z} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{-\partial F/\partial y}{\partial F/\partial z} = \dfrac{-2y}{x+2z-e^z} \end{cases}.$$

§1. Implicit partial differentiation

Suppose we want to analyze the rates of change of z with respect to x & y in a situation where z depends on these variables through an equation such as

$$\underbrace{y^2 + xz + z^2 - e^z}_{=: F(x,y,z)} = 4 \quad \text{level surface of } F$$

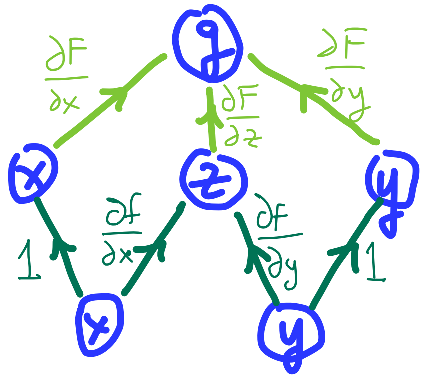
that you can't explicitly solve for z.

Write formally

$$z = f(x,y), \quad g(x,y) := F(x,y,f(x,y))$$

where we insist that g remain constant $\equiv 4$.

Using the "Chain rule diagram" shown,



we obtain

$$\begin{cases} 0 = \frac{\partial g}{\partial x} = \frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial z}\cdot \frac{\partial f}{\partial x} \\ 0 = \frac{\partial g}{\partial y} = \frac{\partial F}{\partial y}\cdot 1 + \frac{\partial F}{\partial z}\cdot \frac{\partial f}{\partial y} \end{cases}$$

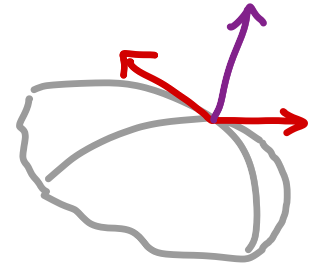
$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-z}{x+2z-e^z} \\ \frac{\partial f}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-2y}{x+2z-e^z}. \end{cases}$$

More informally, $0 = \frac{\partial}{\partial y}(y^2+xz+z^2-e^z)$

$= 2y + x\frac{\partial z}{\partial y} + 2z\frac{\partial z}{\partial y} - e^z\frac{\partial z}{\partial y} \Longrightarrow$

$\frac{\partial z}{\partial y} = \frac{2y}{x+2z-e^z}$.

$$\underbrace{y^2+xz+z^2-e^z}_{=:F(x,y,z)} = 4$$



$z = f(x,y)$, $g(x,y) := F(x,y,f(x,y))$

So at $\vec{a} = \begin{pmatrix} 0 \\ e \\ 2 \end{pmatrix}$ on this level surface we get $\frac{\partial f}{\partial x} = \frac{2}{e^2-4}$, $\frac{\partial f}{\partial y} = \frac{2e}{e^2-4}$

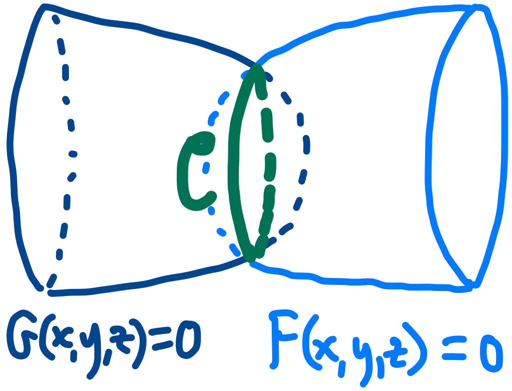
hence tangent vectors $\begin{pmatrix} 1 \\ 0 \\ \frac{2}{e^2-4} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ \frac{2e}{e^2-4} \end{pmatrix}$,

with cross product $\begin{pmatrix} \frac{2}{4-e^2} \\ \frac{2e}{4-e^2} \\ 1 \end{pmatrix}$ normal to the surface. This should be parallel to the gradient $\vec{\nabla} F(\vec{a}) = \begin{pmatrix} z \\ 2y \\ x+2z-e^z \end{pmatrix}(\vec{a})$

$= \begin{pmatrix} 2 \\ 2e \\ 4-e^2 \end{pmatrix}$, and it is. Note the importance of $\frac{\partial F}{\partial z}$ being nonzero: otherwise our equations don't work!

Here is a related computational idea:

given two level surfaces intersecting in a curve $\mathcal{C}$,



let's suppose that on this curve we can parametrise x & y as differentiable functions of z:

$$z \longmapsto \begin{pmatrix} X(z) \\ Y(z) \\ z \end{pmatrix}.$$

Writing $f(z) := F\begin{pmatrix} X(z) \\ Y(z) \\ z \end{pmatrix}$, $g(z) := G\begin{pmatrix} X(z) \\ Y(z) \\ z \end{pmatrix}$, staying on the curve $\mathcal{C}$ means that f and g remain zero:

$$\begin{cases} 0 = f'(z) = F_x X' + F_y Y' + F_z \\ 0 = g'(z) = G_x X' + G_y Y' + G_z \end{cases}$$

$$\Longrightarrow \begin{cases} F_x X' + F_y Y' = -F_z \\ G_x X' + G_y Y' = -G_z \end{cases}$$

i.e. $$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} -F_z \\ -G_z \end{pmatrix},$$

which we can certainly solve for $\begin{pmatrix} X' \\ Y' \end{pmatrix}$ ... provided the matrix has rank 2 !!

So our curve is a level set of

$$\vec{H} = \begin{pmatrix} F \\ G \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

(namely, $\mathcal{C} = \vec{H}^{-1}(\vec{0})$), and it appears that whether the rank of

$$D\vec{H} = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{pmatrix}$$

is maximal governs (locally) whether such a parametrization can exist.

§2. What is a "manifold"?

Instead of giving the general definition, I'll explain here when a level set is a manifold. Let

$$\vec{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^m \ni \vec{c}$$

(more generally you could put an open set $\mathcal{S} \subset \mathbb{R}^n$ here)

be a C^1 map and $M := \vec{F}^{-1}(\vec{c})$ a level set.

You should imagine M as an $(n-m)$-dimensional subset of $\mathbb{R}^n$:

- if $n=3$ and $m=1$, a surface; $m=2$, a curve
- for any n, if $m=1$, M is $(n-1)$-dim'l, and called a *hypersurface*
- if $\vec{F}$ is a linear map, with matrix A, and $\vec{c}=\vec{0}$, then $M = \mathrm{Nul}(A)$!

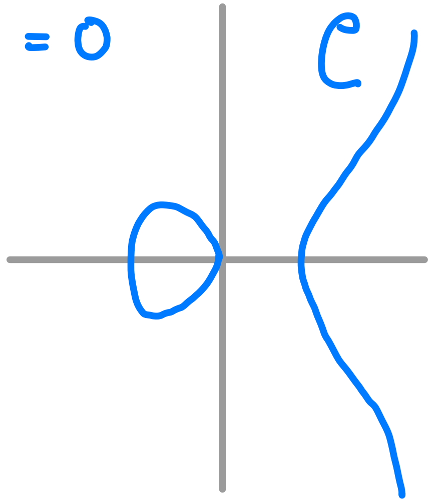
Ex 2/ $\overbrace{y^2 - x^3 + x}^{f(x,y)} = 0$

is a manifold: if

$$Df = (-3x^2+1,\ 2y)$$

is zero, then

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1/\sqrt{3}\\0\end{pmatrix}$ or $\begin{pmatrix}-1/\sqrt{3}\\0\end{pmatrix}$, which are not points of C (i.e. don't satisfy $f(x,y)=0$). //



I pause here to mention that you can easily parametrize the "C" of Ex. 1 by $t \mapsto \begin{pmatrix} t^2-1 \\ t(t^2-1)\end{pmatrix}$, but the "$C$" of Ex. 2 has two connected components, and you can't parametrize either one without the aid of (non-elementary) "elliptic functions".

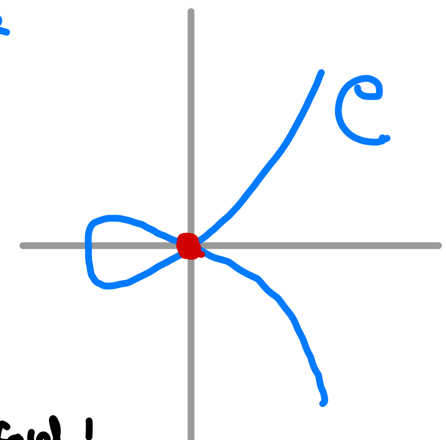
Anyway, here is a provisional

$\boxed{\begin{array}{l}\textbf{Definition: (i) } M \text{ is smooth at } \vec{a} \in M \text{ if } \operatorname{rank}(D\vec{F}(\vec{a})) = m. \\ \text{(ii) } M \text{ is a manifold (or "smooth") if it is smooth at every point.}\end{array}}$

Here are some examples from Shifrin:

Ex 1/ $y^2 = \underbrace{x^3 + x^2}_{x^2(x+1)}$

is a level set (of $f: \mathbb{R}^2 \to \mathbb{R}^1$, with $f(x,y) = y^2 - x^3 - x^2$), but is NOT a manifold!



The problem is at $\vec{0}$, where C is not smooth:

$$Df(\vec{0}) = (f_x(0), f_y(0)) = (0,0)$$

does not have rank 1. //

To show that $\mathcal{C}$ is a smooth curve (i.e. manifold), compute

$$D\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x}(x^2+y^2) & \frac{\partial}{\partial y}(x^2+y^2) & \frac{\partial}{\partial z}(x^2+y^2) \\ \frac{\partial}{\partial x}(x^2+z^2) & \frac{\partial}{\partial y}(x^2+z^2) & \frac{\partial}{\partial z}(x^2+z^2) \end{pmatrix}$$

$$= \begin{pmatrix} 2x & 2y & 0 \\ 2x & 0 & 2z \end{pmatrix}$$

which has rank 2 as long as at least two variables are nonzero.

If this fails, we are at a point of the form $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$; the last two can't happen on $\mathcal{C}$, and $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ can only happen if $a = b$.

So if $a \neq b$, then $\mathcal{C}$ is smooth, and if $a = b$, it isn't. //

Anyway, here is a provisional

<u>Definition</u>: (i) M is <u>smooth</u> at $\vec{a} \in M$ if $\operatorname{rank}(D\vec{F}(\vec{a})) = m$.

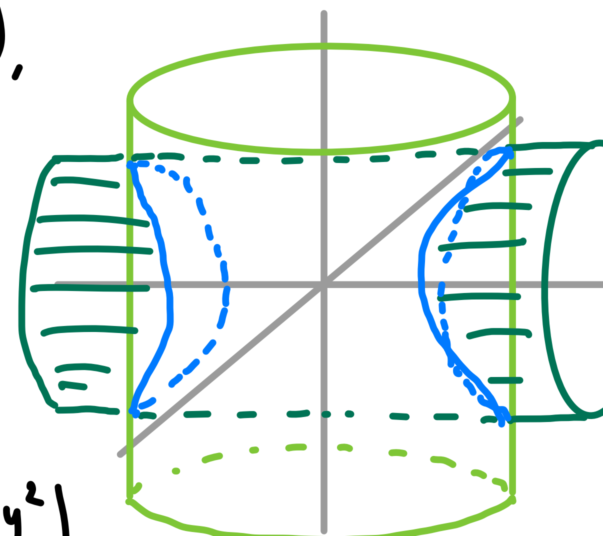
(ii) M is a <u>manifold</u> (or "smooth") if it is smooth at every point.

Here are some examples from Shifrin:

Ex 3/ $\mathcal{C} = \{x^2+y^2 = a^2\} \cap \{x^2+z^2 = b^2\}$

is $\vec{F}^{-1}\begin{pmatrix} a^2 \\ b^2 \end{pmatrix}$, where $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2+y^2 \\ x^2+z^2 \end{pmatrix}.$$



Anyway, here is a provisional

Definition: (i) M is smooth at $\vec{a} \in M$ if $\operatorname{rank}(D\vec{F}(\vec{a})) = m$.

(ii) M is a manifold (or "smooth") if it is smooth at every point.

PROBLEM: Is the surface M defined by

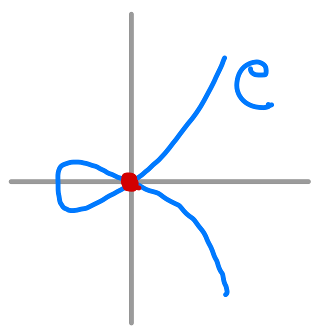
$$y^2 + xz + z^2 - e^z = 4$$

a manifold?

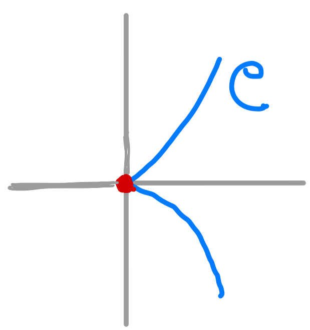
§3. Implicit function theorem

Recall that we got onto this topic (of rank$(D\vec{F}(\vec{a}))$ etc.) because we wanted to parametrize level sets using some of the coordinates by writing the other coordinates as C^1 functions of them.

Singular (i.e. non-smooth) points are kryptonite for this project: something like Ex. 1 doesn't pass the vertical or horizontal line test!



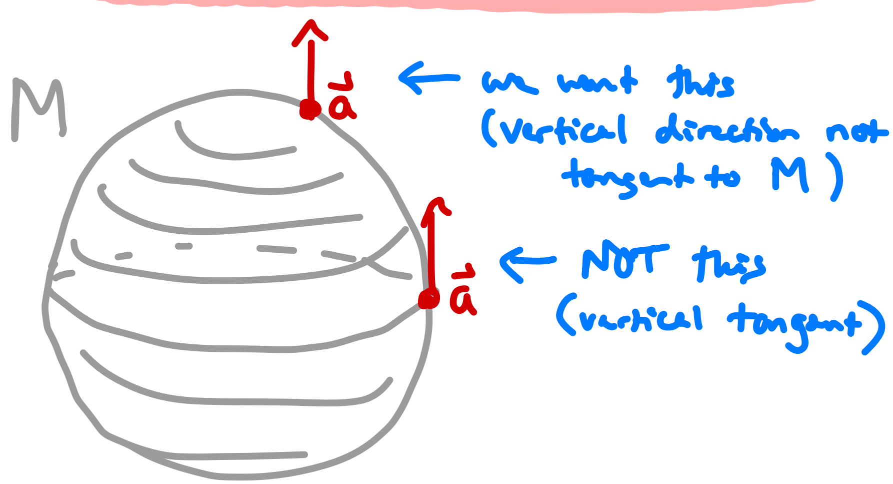
A "cusp" like $y^2 = x^3$ doesn't pass the vertical line test, and $x = y^{2/3}$ isn't a C^1 function of y (since its derivative $\frac{2}{3}y^{-1/3}$ blows up at 0).



Implicit Function Theorem (m=1 version)

Let M be the level set $f^{-1}(0)$ of $f:\mathbb{R}^n \to \mathbb{R}$, and $\vec{a} \in M$ a point at which $\frac{\partial f}{\partial x_n}(\vec{a}) \neq 0$.

INFORMAL STATEMENT: then we can write x_n as a $\mathcal{C}^1$ function of $x_1, \ldots, x_{n-1}$.



The IFT says that, in a neighborhood of a smooth point $\vec{a}$ of a level set M of $\vec{F}:\mathbb{R}^n \to \mathbb{R}^m$ (and hence at any point of a manifold), you can do this: there exist coordinates $x_{i_1}, \ldots, x_{i_{n-m}}$ and $\mathcal{C}^1$ functions by which the other coordinates depend on them in M. If $\vec{F}$ is a linear map, these $x_{i_1}, \ldots, x_{i_{n-m}}$ are nothing but the free variables — and you write the pivot variables in terms of them. So we are after a nonlinear analogue, which the linear version will approximate. Here I'll just describe the case $m=1$, where $M = f^{-1}(0)$ is a hypersurface.

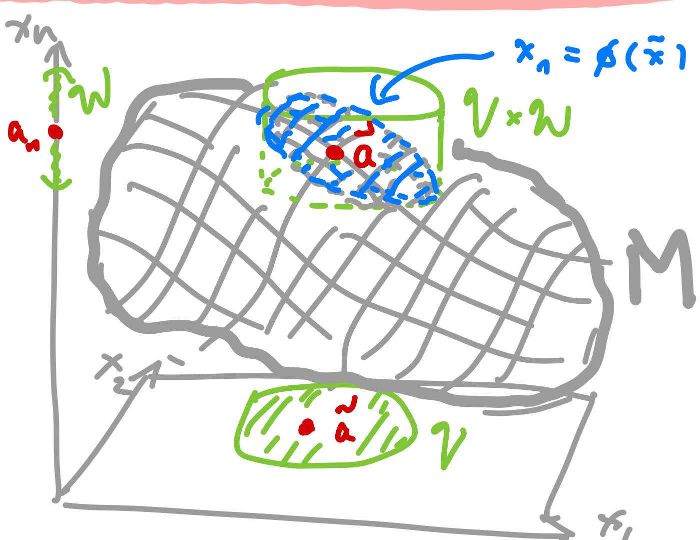
Implicit Function Theorem $\begin{pmatrix} m=1 \\ \text{version} \end{pmatrix}$

Let M be the level set $f^{-1}(0)$ of $f : \mathbb{R}^n \to \mathbb{R}$, and $\vec{a} \in M$ a point at which $\frac{\partial f}{\partial x_n}(\vec{a}) \neq 0$.

Writing $\vec{x} = \begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} \tilde{a} \\ a_n \end{pmatrix}$, i.e. $\tilde{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$,

there are neighborhoods $\mathcal{V} \subset \mathbb{R}^{n-1}$ of $\tilde{a}$ and $\mathcal{W} \subset \mathbb{R}$ of a_n and a C^1 function $\phi : \mathcal{V} \to \mathcal{W}$ s.t. for $\tilde{x} \in \mathcal{V}$ and $x_n \in \mathcal{W}$,

$$\begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix} \in M \iff x_n = \phi(\tilde{x}).$$



As an application, we can relate our notion of tangent plane to M as level surface to the notion of tangent plane to the graph of a function. Define

$$\vec{g} : \mathcal{V} \to \mathbb{R}^n \text{ by } \vec{g}(\tilde{x}) := \begin{pmatrix} \tilde{x} \\ \phi(\tilde{x}) \end{pmatrix},$$

so that $(f \circ \vec{g})(\tilde{x}) = 0 \quad \forall \tilde{x} \in \mathcal{V}$.

The Chain Rule gives

$$[0 \cdots 0] = D(f \circ \vec{g})(\tilde{a}) = Df(\vec{a}) \circ D\vec{g}(\tilde{a})$$

$$= \left(\frac{\partial f}{\partial x_1}(\vec{a}), \ldots, \frac{\partial f}{\partial x_n}(\vec{a}) \right) \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \frac{\partial \phi}{\partial x_1}(\tilde{a}) & \cdots & \frac{\partial \phi}{\partial x_{n-1}}(\tilde{a}) \end{pmatrix}$$

$$= \Big(\ldots, \underbrace{\frac{\partial f}{\partial x_j}(\vec{a}) + \frac{\partial f}{\partial x_n}(\vec{a}) \frac{\partial \phi}{\partial x_j}(\tilde{a})}_{j^{\text{th}} \text{ entry}}, \ldots \Big)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x_j}(\tilde{a}) = \frac{-\partial f / \partial x_j (\vec{a})}{\partial f / \partial x_n (\vec{a})}.$$

Implicit Function Theorem ($m=1$ version)

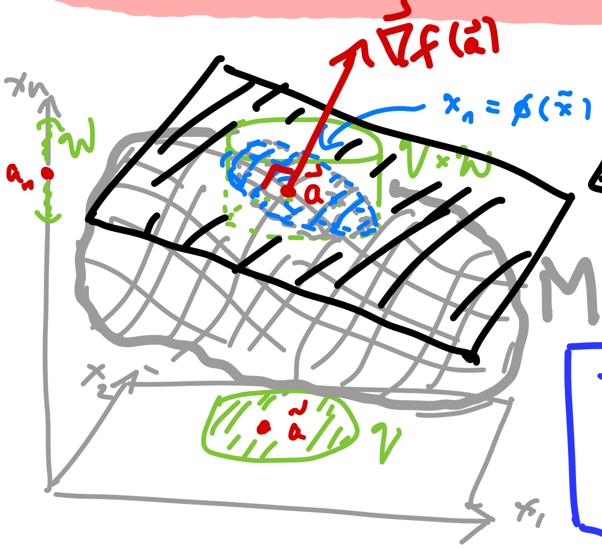
Let M be the level set $f^{-1}(0)$ of $f: \mathbb{R}^n \to \mathbb{R}$, and $\vec{a} \in M$ a point at which $\frac{\partial f}{\partial x_n}(\vec{a}) \neq 0$.

Writing $\vec{x} = \begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} \tilde{a} \\ a_n \end{pmatrix}$,

i.e. $\begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$, $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$

there are neighborhoods $V \subset \mathbb{R}^{n-1}$ of $\tilde{a}$ and $W \subset \mathbb{R}$ of a_n and a C^1 function $\phi: V \to W$ s.t. for $\tilde{x} \in V$ and $x_n \in W$,

$$\begin{pmatrix} \tilde{x} \\ x_n \end{pmatrix} \in M \iff x_n = \phi(\tilde{x}).$$



$$\boxed{\frac{\partial \phi}{\partial x_j}(\tilde{a}) = \frac{-\partial f/\partial x_j(\vec{a})}{\partial f/\partial x_n(\vec{a})}}$$

Now the linear approximation to the graph of $x_n = \phi(\tilde{x})$ at $\vec{a} = \begin{pmatrix} \tilde{a} \\ a_n \end{pmatrix}$ is

$a_n = \phi(\tilde{a})$

$$x_n - a_n = D\phi(\tilde{a})\,(\tilde{x} - \tilde{a})$$

$$= \left(\frac{\partial \phi}{\partial x_1}(\tilde{a}), \dots, \frac{\partial \phi}{\partial x_{n-1}}(\tilde{a})\right) \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_{n-1} - a_{n-1} \end{pmatrix}$$

$$= \sum_{j=1}^{n-1} \frac{\partial \phi}{\partial x_j}(\tilde{a})\,(x_j - a_j)$$

$$= -\sum_{j=1}^{n-1} \frac{\partial f/\partial x_j(\vec{a})}{\partial f/\partial x_n(\vec{a})}\,(x_j - a_j)$$

$$\Longrightarrow \frac{\partial f}{\partial x_n}(\vec{a})\,(x_n - a_n) = -\sum_{j=1}^{n-1} \frac{\partial f}{\partial x_j}(\vec{a})(x_j - a_j)$$

$$\Longrightarrow \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\vec{a})\,(x_j - a_j) = 0$$

$$\Longrightarrow \vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$$

which is exactly our formula for the tangent plane $T_{\vec{a}} M$.