Double Integrals

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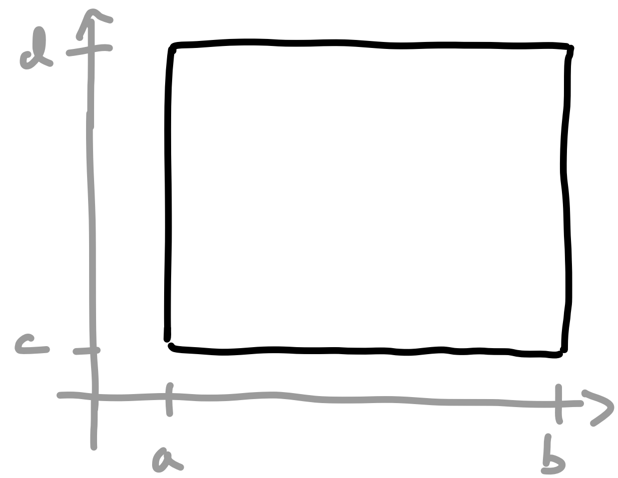
We are finally ready to turn to integral calculus, where we chop a continuous problem (like computing volume, mass, etc.) into small bits and add them up — taking the limit as the "bits" go to zero to get an exact answer.

I will begin with the 2-variable case and do most of the proofs there, so that we have the simpler notation on our side.

§1. Rectangular Double Integrals

Let's start with "step functions" on a rectangle $R = [a,b] \times [c,d]$.



with interiors $\mathring{R}_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$.

$\boxed{\text{Definition: } f : R \to \mathbb{R} \text{ is a step function if, for some partition } P, f \text{ is constant } \equiv c_{ij} \text{ on each } \mathring{R}_{ij}.}$

The integral of f is then defined by

$$\iint_R f \, dA := \sum_{i=1}^{n} \sum_{j=1}^{m} \underset{\substack{\uparrow \\ \text{value of} \\ f \text{ on } R_{ij}}}{c_{ij}} \underbrace{(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{area of } R_{ij}}.$$

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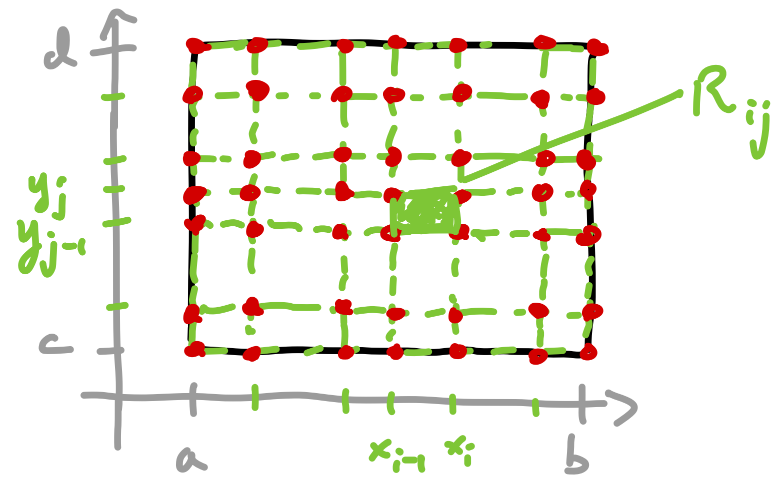


Given $\begin{cases} P_1 = \{x_0 < x_1 < \dots < x_n\}, & x_0 = a,\ x_n = b \\ P_2 = \{y_0 < y_1 < \dots < y_m\}, & y_0 = c,\ y_m = d \end{cases}$

partitions of the two intervals, we call $P = P_1 \times P_2$ a partition of R. It subdivides R into little rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

with interiors $\mathring{R}_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$.

Definition: $f : R \to \mathbb{R}$ is a step function if, for some partition P, f is constant $\equiv c_{ij}$ on each $\mathring{R}_{ij}$.

The integral of f is then defined by

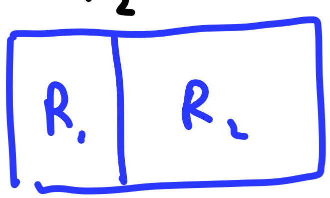
$$\iint_R f\, dA := \sum_{i=1}^{n} \sum_{j=1}^{m} \underset{\substack{\uparrow \\ \text{value of} \\ f \text{ on } R_{ij}}}{c_{ij}} \underbrace{(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{area of } R_{ij}}.$$

PROPERTIES: for step functions f & g,

- $\iint_R (af + bg)\, dA = a \iint_R f\, dA + b \iint_R g\, dA$
- $\iint_R f\, dA = \iint_{R_1} f\, dA + \iint_{R_2} f\, dA$

  if $R = R_1 \cup R_2$

  with $R_1 \cap R_2$ a segment



- $\iint_R f\, dA \le \iint_R g\, dA$ if $f \le g$
- $\iint_R f\, dA = \int_c^d \left(\int_a^b f\binom{x}{y}\, dx \right) dy$.

Now let $f : R \to \mathbb{R}$ be a (non-step) function; assume $|f| \le M$. Being constant, M & $-M$ are step functions, and $-M \le f \le M$.

So $S := \left\{ \iint_R s\, dA \mid s \text{ step}, s \le f \right\}$ is bounded above by $\iint_R M\, dA = M \cdot \text{area}(R)$, while $-M \cdot \text{area}(R)$ gives a lower bound on the set $T := \left\{ \iint_R t\, dA \mid t \text{ step}, t \ge f \right\}$.

Therefore $\underline{I} := \sup(S)$ and $\overline{I} := \inf(T)$ exist!

Moreover, since for any $\overset{\text{step}}{t} \geq f$

$$\iint_R s\,dA \leq \iint_R t\,dA \quad \forall \overset{\text{step}}{s} \leq f,$$

we have $\underline{I} \leq \iint_R t\,dA \quad (\forall t)$

$\Longrightarrow \quad \underline{I} \leq \overline{I}.$

Definition: If $\underline{I} = \overline{I}$ then f is integrable on R, and $\iint_R f\,dA$ is defined to be this common value.

This double integral satisfies the same properties as the version for step functions: e.g., if f and g are integrable on R, then so is $f+g$, and

$$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA.$$

The proof is by noticing that we can choose step functions $s_f \leq f \leq t_f$ & $s_g \leq g \leq t_g$ that make, in

$$\iint (s_f+s_g)\,dA \leq \iint f\,dA + \iint g\,dA \leq \iint (t_f+t_g)\,dA,$$

the difference RHS − LHS arbitrarily small. Since $s_f + s_g \leq f+g \leq t_f+t_g$, this makes the middle term $\iint_R (f+g)\,dA$ by definition!

But... how to compute it?!

Proof of Fubini: Let $s \le f \le t$, with s & t step functions. For each y,

$$\underbrace{\int_a^b s\binom{x}{y}\,dx \le \int_a^b f\binom{x}{y}\,dx}_{} \le \int_a^b t\binom{x}{y}\,dx$$

hence

$$\int_c^d\int_a^b s\,dx\,dy \le \int_c^d \mathcal{A}(y)\,dy \le \int_c^d\int_a^b t\,dx\,dy$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_R s\,dA \qquad\qquad\qquad\qquad \int_R t\,dA$$

Since s and t are arbitrary,

sup inf

$$\underline{I} \le \int_c^d \mathcal{A}(y)\,dy \le \overline{I}$$

$\underline{I} = \iint_R f\,dA = \overline{I}$ by assumption. $\square$

Definition: If $\underline{I} = \overline{I}$ then f is integrable on R, and $\iint_R f\,dA$ is defined to be this common value.

But... how to compute it?!

FUBINI'S THEOREM: If $f: R \to \mathbb{R}$ is integrable on R, $f\binom{x}{y_0}$ is integrable on $[a,b]$ for each $y_0 \in [c,d]$, and $\mathcal{A}(y) := \int_a^b f\binom{x}{y}\,dx$ is integrable on $[c,d]$, then

$$\iint_R f\,dA = \int_c^d \mathcal{A}(y)\,dy.$$

N.B.: (i) We can also reverse the rôles of x & y.

(ii) $\int_c^d\left(\int_a^b f\binom{x}{y}\,dx\right)dy$ is called an iterated integral.

Ex/ Find $\iint_{[1,2]\times[0,\pi]} y\sin(xy)\,dA =$

$$\int_1^2\left(\int_0^\pi y\sin(xy)\,dy\right)dx = \leftarrow$$

$\int$ by parts: $u=y$, $du=dy$, $dv=\sin(xy)\,dy$, $v=\frac{-1}{x}\cos(xy)$

$$\int_1^2\left[-\frac{y}{x}\cos(xy)\Big|_{y=0}^{\pi} + \int_0^\pi \frac{1}{x}\cos(xy)\,dy\right]dx =$$

$$\int_1^2\left[-\frac{\pi}{x}\cos(\pi x)+\frac{1}{x^2}\sin(\pi x)\right]dx = \leftarrow$$

$\int$ by parts on 1st term: $u=-\frac{1}{x}$, $dv=\pi\cos(\pi x)\,dx$ etc.

$$-\frac{1}{x}\sin(\pi x)\Big|_1^2 - \cancel{\int_1^2\frac{1}{x^2}\sin(\pi x)\,dx} + \cancel{\int_1^2\frac{1}{x^2}\sin(\pi x)\,dx} =$$

$$-\frac{1}{x}\sin(\pi x)\Big|_1^2 = -\frac{1}{2}\sin 2\pi + \sin\pi = 0.$$

(A lot of work for that 0...)

(alternatively)

$$\int_0^\pi\left(\int_1^2 y\sin(xy)\,dx\right)dy =$$

$$\int_0^\pi\left[-\cos(xy)\right]_{x=1}^2\,dy = \int_0^\pi(\cos y - 2\cos 2y)\,dy$$

$$= \left[\sin y - \frac{1}{2}\sin 2y\right]_0^\pi = 0.$$

Moral: the order you choose can make a big difference (for the work, not the answer).

Problem: Use Fubini's Theorem to compute

$$\int_0^{\sqrt{3}}\left(\int_0^1 \frac{8x}{(x^2+y^2+1)^2}\,dy\right)dx.$$

Sometimes $f\binom{x}{y}$ breaks up as $F(x)G(y)$.

For example, $\frac{y}{1+x^2}$ or $xy\,e^{x^2+y^2}$.

In this case, the double integral itself factors:

$$\iint_R f\binom{x}{y}\,dA = \int_a^b\left(\int_c^d \overbrace{F(x)}^{\text{constant w.r.t. } y} G(y)\,dy\right)dx$$

$$= \int_a^b\left(F(x)\underbrace{\int_c^d G(y)\,dy}_{\text{constant w.r.t. } x}\right)dx$$

$$= \left(\int_c^d G(y)\,dy\right)\left(\int_a^b F(x)\,dx\right).$$

Ex/ $\displaystyle\int_{[0,1]\times[0,1]} xy\,e^{x^2+y^2}\,dA \leftarrow \underbrace{x e^{x^2}\cdot y e^{y^2}}$

$$= \left(\int_0^1 x e^{x^2}\,dx\right)^2 = \left(\left[\frac{1}{2}e^{x^2}\right]_0^1\right)^2$$

$$= \left(\frac{1}{2}(e-1)\right)^2 = \frac{(e-1)^2}{4}.$$

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Problem: Use Fubini's Theorem to compute

$$I := \int_0^{\sqrt{3}}\left(\int_0^1 \frac{8x}{(x^2+y^2+1)^2}\,dy\right)dx.$$

Obviously we want to avoid the inner integral; using Fubini twice lets us write

$$I = \int_{[0,\sqrt{3}]\times[0,1]} \frac{8x}{(x^2+y^2+1)^2}\,dA = \int_0^1\left(\int_0^{\sqrt{3}} \frac{8x}{(x^2+y^2+1)^2}\,dx\right)dy$$

$$= \int_0^1 \left[\frac{-4}{x^2+y^2+1}\right]_{x=0}^{\sqrt{3}} dy$$

$$= \int_0^1 \left(\underbrace{\frac{-4}{4+y^2}}_{\frac{-1}{1+(y/2)^2}} + \frac{4}{1+y^2}\right) dy$$

$$= \left[-2\arctan\left(\frac{y}{2}\right) + 4\arctan(y)\right]_0^1$$

$$= \left(-2\arctan\left(\frac{1}{2}\right) + 4\arctan(1)\right) - (0+0)$$

$$= \pi - 2\arctan\frac{1}{2}.$$

§ 2. Continuity ⇒ integrability

Lemma: Let $\mathcal{S} \subset \mathbb{R}^n$ be compact, $f : \mathcal{S} \to \mathbb{R}$ continuous. Then f is uniformly continuous: given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|\vec{x} - \vec{y}\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{y})| < \epsilon$.

Proof: If f weren't uniformly continuous, there would exist $\epsilon_0 > 0$ & $\vec{x}_m, \vec{y}_m \in \mathcal{S}$ with $\|\vec{x}_m - \vec{y}_m\| < \frac{1}{m}$ and $|f(\vec{x}_m) - f(\vec{y}_m)| \geq \epsilon_0$ for all $m \in \mathbb{N}$. By compactness, $\exists$ subsequence $\vec{x}_{m_k} \to \vec{a} \in \mathcal{S}$, and then $\vec{y}_{m_k} \to \vec{a}$ too.

By continuity of f at $\vec{a}$, $\exists\, \delta_0 > 0$ s.t. $\|\vec{x} - \vec{a}\| < \delta_0 \Rightarrow |f(\vec{x}) - f(\vec{a})| < \epsilon_0/2$.

So for k suff. large, $\|\vec{x}_{m_k} - \vec{a}\|, \|\vec{y}_{m_k} - \vec{a}\| < \delta_0$

$$\Rightarrow |f(\vec{x}_{m_k}) - f(\vec{y}_{m_k})| \leq |f(\vec{x}_{m_k}) - f(\vec{a})| + |f(\vec{y}_{m_k}) - f(\vec{a})| < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0,$$

a contradiction! $\square$

Theorem: Let $R \subset \mathbb{R}^2$ be a rectangle and $f : R \to \mathbb{R}$ be continuous. Then f is integrable.

Proof: R compact + f C^0 $\Rightarrow$ f bounded $\Rightarrow$ $\overline{I}$ & $\underline{I}$ exist (by argument in §1).

Lemma $\Rightarrow$ f uniformly C^0 $\Rightarrow$ for each $\epsilon > 0$, $\exists$ partition $\mathcal{P}$ of R s.t. on each R_{ij} the difference of the maximum $M_{ij}(f)$ & minimum $m_{ij}(f)$ is $< \epsilon$.

Define $s|_{\mathring{R}_{ij}} := m_{ij}(f)$ & $t|_{\mathring{R}_{ij}} := M_{ij}(f)$, so that $s \leq f \leq t$. Then

$$\underbrace{\sum_{i,j} m_{ij}(f) \cdot \text{area}(R_{ij})} = \iint_R s \, dA \leq \underline{I}$$

$$\leq \overline{I} \leq \iint_R t \, dA = \underbrace{\sum_{i,j} M_{ij}(f) \cdot \text{area}(R_{ij})}$$

while RHS $-$ LHS $< \epsilon \cdot \text{area}(R)$ $\Rightarrow$

$0 \leq \overline{I} - \underline{I} < \epsilon \cdot \text{area}(R)$ $\xRightarrow{\epsilon \text{ arbitrary}}$ $\overline{I} = \underline{I}$

$\Rightarrow f$ integrable. $\square$

Problem: Suppose $f: R \to \mathbb{R}$ is nonnegative, continuous, and positive at some point $\vec{a} \in R$. Prove that

$$\int_R f \, dA > 0.$$

By continuity, $\exists\, \delta > 0$ s.t. $\|\vec{x} - \vec{a}\| < \delta$
$\Rightarrow |f(\vec{x}) - f(\vec{a})| < \frac{f(a)}{2} \Rightarrow f(x) \geq \frac{f(a)}{2}$.
Choose a partition P s.t. the subrectangle R_0 containing $\vec{a}$ has diameter $< \delta$, and pick the step function s which is $\frac{f(a)}{2}$ on R_0 and 0 on $R \setminus R_0$. Then $f \geq s$ (on all of R)

$$\Rightarrow \int_R f \, dA \geq \int_R s \, dA = \frac{f(a)}{2} \cdot \text{area}(R_0) > 0$$

§3. Nonrectangular Double Integrals

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Let $\mathcal{S}$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle R:

If $f\begin{pmatrix}x\\y\end{pmatrix}$ is a function on $\mathcal{S}$, define a function on R by

$$\tilde{f}\begin{pmatrix}x\\y\end{pmatrix} := \begin{cases} f\begin{pmatrix}x\\y\end{pmatrix} & \text{if } \begin{pmatrix}x\\y\end{pmatrix} \in \mathcal{S} \\ 0 & \text{otherwise}. \end{cases}$$

Definition: f is integrable on $\mathcal{S}$ if $\tilde{f}$ is integrable on R, in which case

$$\iint_{\mathcal{S}} f \, dA := \iint_R \tilde{f} \, dA.$$

In particular, if the constant function 1 on S is integrable, then we can define

$$\text{area}(S) := \iint_S 1 \, dA.$$

In order to have even this integral on S defined, we need to make sure that its boundary isn't too bad:

Definition: A subset $D \subset R$ has content zero if for each $\epsilon > 0$, there exists a finite union of rectangles of total area $< \epsilon$ and containing D.

Theorem: Let $f : R \to \mathbb{R}$ be bounded, and continuous on $R \setminus D$, where D has content zero. Then f is integrable.

Proof: Let $\delta > 0$ be given, and choose a partition $\mathcal{P}$ of R so that amongst the R_{ij} are some rectangles as described in the Definition.

Define step functions s & t as in the last proof; except on the R_{ij} covering D we set $s = -M$ and $t = M$ (where $|f| \leq M$ on R).

Arguing as in the last proof, we get

$$0 \leq \overline{I} - \underline{I} < 2M\epsilon + \epsilon \cdot \text{area}(R)$$

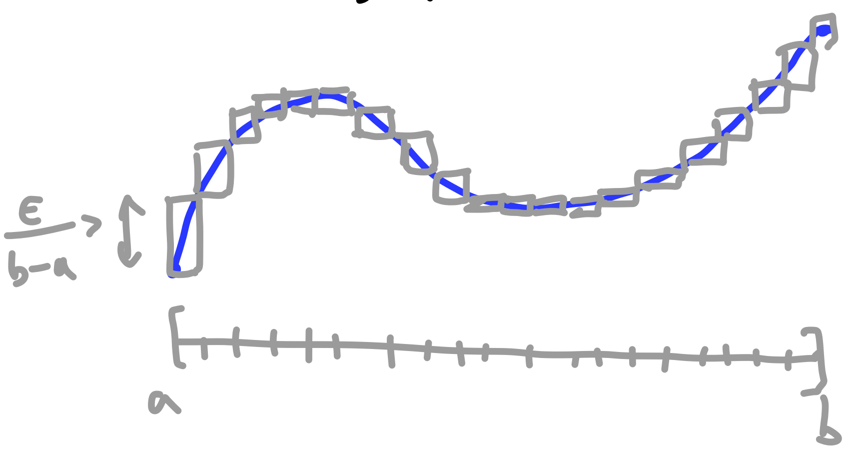
and taking $\epsilon \to 0$ we get $\overline{I} = \underline{I}$. $\square$

Shifrin calls a subset $\Omega \subset \mathbb{R}^n$ a region if it is the closure of a bounded open subset and $\partial\Omega$ has content zero. An immediate corollary is that continuous functions on regions are integrable; so e.g. $\text{area}(\Omega)$ is defined.

But what sorts of sets are regions?

Lemma: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

Proof: Let Γ = graph of $\varphi : [a,b] \to \mathbb{R}$ C^0. Fix $\epsilon > 0$. Since φ is uniformly continuous, we can partition $[a,b]$ into finitely many intervals $I_i = [x_{i-1}, x_i]$ so that the "max−min" of φ is $< \frac{\epsilon}{b-a}$ on I_i.



The sum of the areas of the boxes in the picture is therefore $< \epsilon$. $\square$

Corollary: Sets of

Type I: $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x) \right\}$

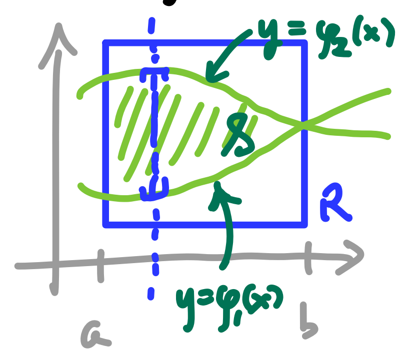
and

Type II: $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y) \right\}$

(with ϕ_i, ψ_i continuous) are regions.

So continuous functions on them are integrable.

This lets us officially use Fubini to do what you did in Calc III: given



$f : S \to \mathbb{R}$ C^0, extend f by 0 on $R \setminus S$ to get $\tilde{f}$; then

$$\iint_S f\,dA = \iint_R \tilde{f}\,dA \underset{\text{Fubini}}{=} \int_a^b \left(\int_c^d \tilde{f}\begin{pmatrix} x \\ y \end{pmatrix} dy \right) dx$$

$$= \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f\begin{pmatrix} x \\ y \end{pmatrix} dy \right) dx.$$

We'll pick up here next time.