Multiple Integrals 18

Last time we defined a bounded function f on a rectangle $R \subset \mathbb{R}^2$ to be integrable if we can make

$$\int_R t\, dA - \int_R s\, dA \quad (\geq 0)$$

arbitrarily small, where $s \leq f \leq t$ are step functions. In that case we defined

$$\int_R f\, dA := \sup_{s \leq f} \left\{ \int_R s\, dA \right\} = \inf_{t \geq f} \left\{ \int t\, dA \right\}.$$

We showed that all bounded functions whose discontinuity set $D \subset R$ has content zero (e.g. piecewise smooth curves) are integrable. This opened the door to defining, for f a continuous function on a region $\Omega \subset R$,

$$\int_\Omega f\, dA := \int_R \tilde{f}\, dA$$

where $\tilde{f}$ is the (integrable!) extension-by-zero of f to all of R.

Integrals over R can be computed by

$$\boxed{\begin{array}{l}\text{FUBINI'S THEOREM: If } f: R \to \mathbb{R} \text{ is integrable on } R,\ f\begin{pmatrix} x \\ y_0 \end{pmatrix} \text{ is integrable} \\ \text{on } [a,b] \text{ for each } y_0 \in [c,d], \text{ and } a(y) := \int_a^b f\begin{pmatrix} x \\ y \end{pmatrix} dx \text{ is integrable on} \\ [c,d], \text{ then} \\ \qquad \iint_R f\, dA = \int_c^d a(y)\, dy.\end{array}}$$

Specifically, we had that sets of

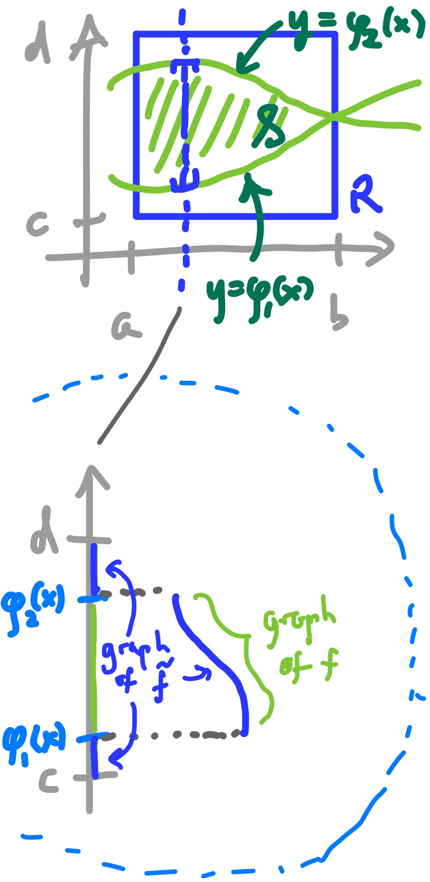
Type I: $\left\{\binom{x}{y} \;\middle|\; a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\right\}$

and

Type II: $\left\{\binom{x}{y} \;\middle|\; c\le y\le d,\ \psi_1(y)\le x\le\psi_2(y)\right\}$

(with ϕ_i, ψ_i continuous) are regions.

So continuous functions on them are integrable, and we may use Fubini to compute their integrals: given



$f: S\to\mathbb{R}$ C^0, extend f by 0 on $R\setminus S$ to get $\tilde f$; then

$$\iint_S f\,dA = \iint_R \tilde f\,dA$$

$$\overset{\text{Fubini:}}{=} \int_a^b\Big(\underbrace{\int_c^d \tilde f\binom{x}{y}dy}\Big)dx$$

$$= \int_a^b\Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f\binom{x}{y}dy\Big)dx.$$

$$\int_\Omega f\,dA := \int_R \tilde f\,dA$$

where $\tilde f$ is the (integrable!) extension-by-zero of f to all of R.

Integrals over R can be computed by

FUBINI'S THEOREM: If $f: R\to\mathbb{R}$ is integrable on R, $f\binom{x}{y_0}$ is integrable on $[a,b]$ for each $y_0\in[c,d]$, and $\mathcal{A}(y) := \int_a^b f\binom{x}{y}dx$ is integrable on $[c,d]$, then

$$\iint_R f\,dA = \int_c^d \mathcal{A}(y)\,dy.$$

And when we left off we were starting to apply this to compute integrals over nonrectangular regions of "type I" and "type II".

Specifically, we had that sets of

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and

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(with φ_i, ψ_i continuous) are regions.

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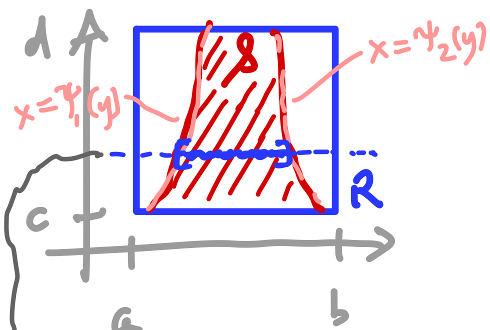
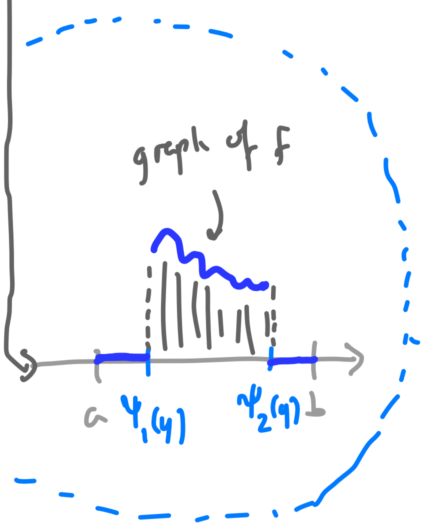


$f : S \to \mathbb{R}$ C^0, extend f by 0 on $R \setminus S$ to get $\tilde{f}$; then

$$\iint_S f\,dA = \iint_R \tilde{f}\,dA$$

$$\overset{\text{Fubini}}{=} \int_c^d \left(\int_a^b \tilde{f}\begin{pmatrix} x \\ y \end{pmatrix} dx \right) dy$$

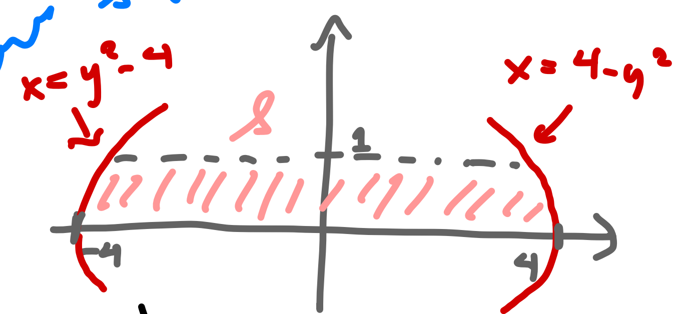
$$= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f\begin{pmatrix} x \\ y \end{pmatrix} dx \right) dy.$$



§1. More on double integrals

The first main thing here is to do some examples.

Ex/ Compute $\int_S (y+1)\,dA$



$$\int_0^1 \left(\int_{y^2-4}^{4-y^2} (y+1)\,dx \right) dy$$

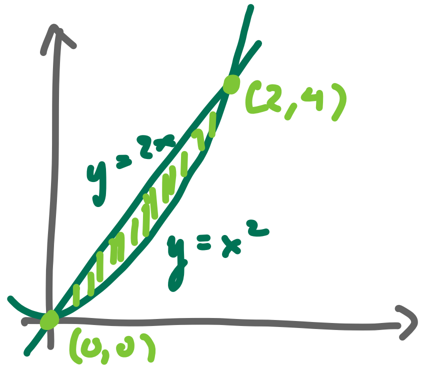
constant w.r.t. x

$$= \int_0^1 \Big[x(y+1) \Big]_{x=y^2-4}^{x=4-y^2} dy$$

$$= \int_0^1 \left(-y^2+4-(y^2-4)\right)(y+1)\,dy$$

$$= \cdots = \frac{65}{6}. \quad //$$

PROBLEM. Find $\int_S (8x+10y)\,dA$, where S is the region between the graphs of $y=x^2$ and $y=2x$.



$$\int_0^2 \left(\int_{x^2}^{2x} (8x+10y)\,dy \right) dx$$

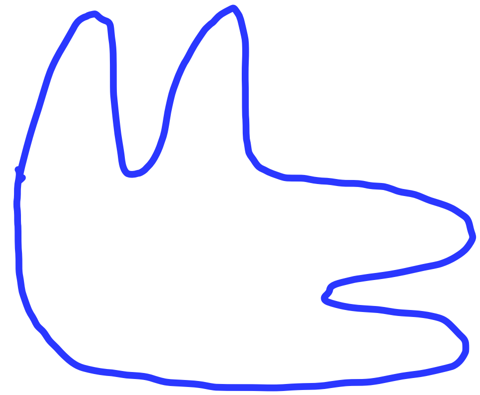
$$= \int_0^2 \left[8xy + 5y^2 \right]_{y=x^2}^{2x} dx$$

$$= \int_0^2 \left((16x^2+20x^2) - (8x^3+5x^4) \right) dx$$

$$= \int_0^2 \left(-5x^4 - 8x^3 + 36x^2 \right) dx$$

$$= \cdots = 32.$$

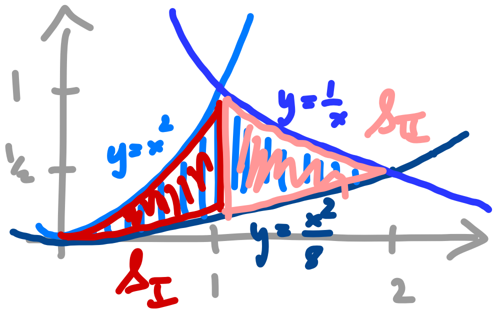
What if S is more complicated?



(That is, neither type I nor type II.)

Draw it:

Divide it:



$$\int_{\mathcal{S}} 1\, dA = \int_{\mathcal{S}_{\mathrm{I}}} 1\, dA + \int_{\mathcal{S}_{\mathrm{II}}} 1\, dA$$

(by the way, this computes area of $\mathcal{S}$)

$$= \int_0^1 \int_{x^2/8}^{x^2} 1\, dy\, dx + \int_1^2 \int_{x^2/8}^{1/x} 1\, dy\, dx$$

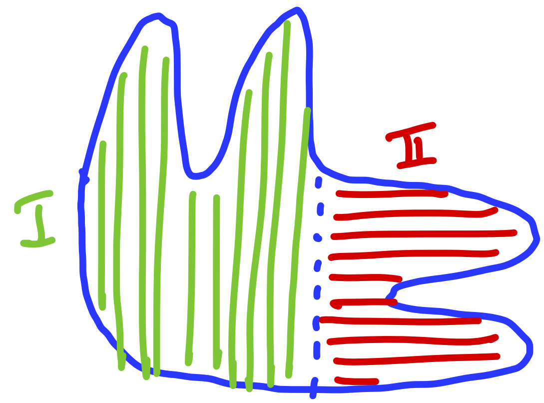
$$= \int_0^1 \left(x^2 - \frac{x^2}{8}\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{x^2}{8}\right) dx$$

$$= \cdots$$

$$= \log(2).$$

//

What if $\mathcal{S}$ is more complicated?



$\mathcal{S} = \mathcal{S}_{\mathrm{I}} \cup \mathcal{S}_{\mathrm{II}}$

(and $\mathcal{S}_{\mathrm{I}} \cap \mathcal{S}_{\mathrm{II}}$ is a segment)

(That is, neither type I nor type II.)

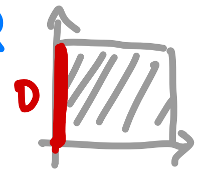
Split it into subsets and write $\int_{\mathcal{S}} = \int_{\mathcal{S}_{\mathrm{I}}} + \int_{\mathcal{S}_{\mathrm{II}}}$.

More generally, you can use this technique (of breaking the region) to compute integrals which are of type I or II:

Ex/ Find $\int_{\mathcal{S}} 1\, dA$, where $\mathcal{S}$ is in the first quadrant, bounded by $y = x^2$, $y = \frac{x^2}{8}$, and $y = \frac{1}{x}$.

Ex/ Find $\int_{[0,1]\times[0,1]} f\binom{x}{y}\,dA$, where

$$f\binom{x}{y} := \begin{cases} 1 & x=0 \text{ \& } y\in\mathbb{Q} \\ 0 & \text{otherwise}. \end{cases}$$



First, f is integrable b/c its discontinuity set is contained in a set of content zero, the line segment D shown.

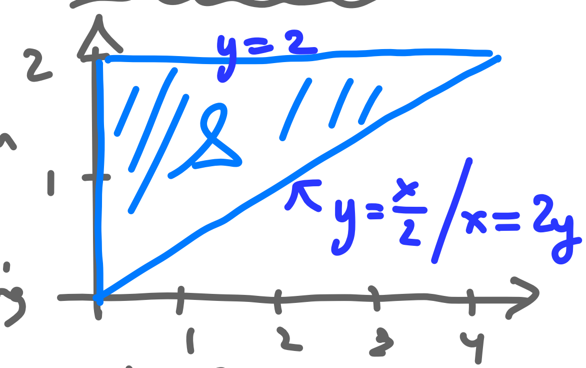
We cannot use Fubini in the form $\int_0^1\left(\int_0^1 f\binom{x}{y}\,dy\right)dx$, b/c $f\binom{0}{y}$ is non-integrable. However, we can use it in the form $\int_0^1\left(\int_0^1 f\binom{x}{y}\,dx\right)dy$, and

$\int_0^1 f\binom{x}{y}\,dx = 0$ for each y (why?);

so $\int_{[0,1]^2} f\,dA = 0$. //

PROBLEM: Find $\int_0^4\left(\int_{x/2}^2 e^{y^2}\,dy\right)dx. = I$

Well, of course you can't antidifferentiate e^{x^2}. But you also can't just "exchange the integrals" — you'd end up with outer limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration (by using Fubini twice):



$$I = \int_{\mathcal{R}} e^{y^2}\,dA = \int_0^2\left(\int_0^{2y} e^{y^2}\,dx\right)dy$$

$$= \int_0^2 2y\,e^{y^2}\,dy = \left[e^{y^2}\right]_0^2$$

$$= e^4 - 1.$$

If moreover $\frac{\partial f}{\partial x}$ is assumed C^0, then $\phi(t) := \int_c^d \frac{\partial f}{\partial x}\binom{t}{y}\,dy$ is defined, and by Fubini

$$\int_a^x \phi(t)\,dt = \int_a^x \int_c^d \frac{\partial f}{\partial x}\binom{t}{y}\,dy\,dt$$
$$= \int_c^d \int_a^x \frac{\partial f}{\partial x}\binom{t}{y}\,dt\,dy$$
$$= \int_c^d \left(f\binom{x}{y} - f\binom{a}{y}\right)dy$$
$$= F(x) - F(a)$$

$$\Rightarrow F'(x) = \frac{d}{dx}\int_a^x \phi(t)\,dt = \phi(x)$$

$$\Rightarrow \boxed{\frac{d}{dx}\int_c^d f\binom{x}{y}\,dy = \int_c^d \frac{\partial f}{\partial x}\binom{x}{y}\,dy},$$

that is, we can "differentiate under the integral sign".

Finally, a bit of theory: if $f \le g$ on $\mathcal{S}$, then $\tilde f \le \tilde g$ on R hence the step function $0 \le \tilde g - \tilde f$ on $R \Rightarrow$

$$0 \le \underline{I} = \int_R (\tilde g - \tilde f)\,dA = \int_R \tilde g\,dA - \int_R \tilde f\,dA$$

$\Rightarrow \int_{\mathcal{S}} f\,dA \le \int_{\mathcal{S}} g\,dA$. Hence

$$-|f| \le f \le |f| \Rightarrow -\int |f|\,dA \le \int f\,dA \le \int |f|\,dA$$

$\Rightarrow \left|\int_{\mathcal{S}} f\,dA\right| \le \int_{\mathcal{S}} |f|\,dA$ in general.

Now suppose $f\binom{x}{y}$ is a continuous (hence uniformly so) function on $R = [a,b]\times[c,d]$. Pick $\delta > 0$ so that

$$\left\|\binom{x}{y} - \binom{x_0}{y_0}\right\| < \delta \Rightarrow \left|f\binom{x}{y} - f\binom{x_0}{y_0}\right| < \frac{\epsilon}{c-d}.$$

Then $F(x) := \int_c^d f\binom{x}{y}\,dy$ is also C^0 since $|F(x) - F(x_0)| = \left|\int_c^d \left(f\binom{x}{y} - f\binom{x_0}{y}\right)dy\right|$

$$\le \int_c^d \underbrace{\left|f\binom{x}{y} - f\binom{x_0}{y}\right|}_{<\epsilon/(c-d)}\,dy < \epsilon.$$

Theorem 1: If ∂S has content zero (can be covered by boxes of arbitrarily small total volume), and f is C^0, then f is integrable.

Theorem 2 (Fubini): If everything is integrable,

$$\int_R \tilde{f}\, dV = \int_{a_n}^{b_n} \cdots \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \tilde{f}(x_1, \ldots, x_n)\, dx_1 \right) dx_2 \right) \cdots dx_n$$

(or in any other order).

The proofs are the same. We also have

Theorem 3: If $X \subset \mathbb{R}^m$ is compact, $U \subset \mathbb{R}^m$ is open and contains X, and $\vec{\phi}: U \to \mathbb{R}^n$ $(n > m)$ is C^1, then $\vec{\phi}(X)$ has content zero.

So all manifolds in $\mathbb{R}^n$ (of dim. $< n$) have content zero; the same goes for graphs of C^0 functions over compact sets.

§2. Multiple Integrals

These work just like double integrals: let $S \subset \mathbb{R}^n$ be a bounded set, $f: S \to \mathbb{R}$ a function on it, and $\tilde{f}: R \to \mathbb{R}$ its extension by 0 to some enclosing box $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Subdividing R into $R_{\underline{i} = (i_1, \ldots, i_n)}$ $\begin{pmatrix} 1 \le i_1 \le m_1 \\ \vdots \\ 1 \le i_n \le m_n \end{pmatrix}$ we define step functions by $s|_{\mathring{R}_{\underline{i}}} \equiv c_{\underline{i}}$, their integrals by $\int_R s\, dV := \sum_{\underline{i}} c_{\underline{i}} \underbrace{\mathrm{vol}(R_{\underline{i}})}_{\text{n-volume of box}}$, and define

$$\underset{(\text{on } S)}{f \text{ integrable}} \iff \underset{(\text{on } R)}{\tilde{f} \text{ integrable}}$$

$$\overset{\text{def.}}{\iff} \sup\left\{ \int_R s\, dV \;\middle|\; s \le \tilde{f} \text{ step} \right\} = \inf\left\{ \int_R t\, dV \;\middle|\; t \ge \tilde{f} \text{ step} \right\}$$

In which case this common value <u>defines</u> $\int_S f\, dV$.

Theorem 1: If ∂S has content zero (can be covered by boxes of arbitrarily small total volume), and f is C^0, then f is integrable.

Theorem 2 (Fubini): If everything is integrable,

$$\int_R \tilde{f}\, dV = \int_{a_n}^{b_n} \cdots \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \tilde{f}(x_1, \dots, x_n)\, dx_1 \right) dx_2 \right) \cdots dx_n$$

(or in any other order).

The proofs are the same. We also have

Theorem 3: If $X \subset \mathbb{R}^m$ is compact, $U \subset \mathbb{R}^m$ is open and contains X, and $\vec{\phi}: U \to \mathbb{R}^n$ $(n > m)$ is C^1, then $\vec{\phi}(X)$ has content zero.

So all manifolds in $\mathbb{R}^n$ (of dim. $< n$) have content zero; the same goes for graphs of C^0 functions over compact sets.

Proof of Theorem 3: We may assume $X \subset [0,1]^m$. Subdivide $[0,1]^m$ into N^m cubes of volume $(1/N)^m$, and let K be the union of cubes that meet X. By taking N large, we may assume $K \subset U$.

Since K is compact, $D\vec{\phi}$ is uniformly continuous on K; so $\|D\vec{\phi}\| \leq M$ there. By the mean value inequality, for $\vec{a}, \vec{b}$ in the same "little cube" we have

$$\|\vec{\phi}(\vec{b}) - \vec{\phi}(\vec{a})\| \leq M \|\vec{b} - \vec{a}\| \leq \frac{M\sqrt{m}}{N}.$$

Pick any point in the image of this little cube and draw a cube centered about it with side length $\frac{2M\sqrt{m}}{N} =: \frac{k}{N}$; it contains the image of the little cube & has volume $\frac{k^n}{N^n}$. Their union contains $\vec{\phi}(X)$ and has total volume $\leq$

$$N^m \cdot \frac{k^n}{N^n} = \frac{k^n}{N^{n-m}} \to 0 \text{ as } N \to \infty. \quad \square$$

Theorem 1: If ∂S has content zero (can be covered by boxes of arbitrarily small total volume), and f is C^0, then f is integrable.

Theorem 2 (Fubini): If everything is integrable,

$$\int_R \tilde{f}\, dV = \int_{a_n}^{b_n} \cdots \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \tilde{f}(x_1, \ldots, x_n)\, dx_1 \right) dx_2 \right) \cdots dx_n$$

(or in any other order).

The proofs are the same. We also have

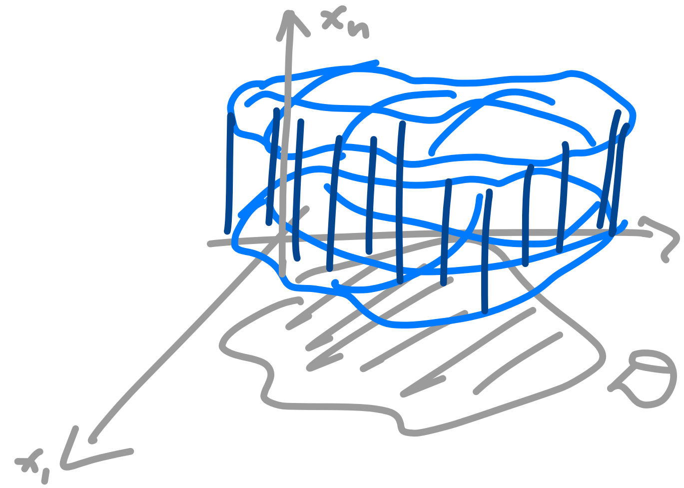
Theorem 3: If $X \subset \mathbb{R}^m$ is compact, $U \subset \mathbb{R}^m$ is open and contains X, and $\vec{\phi} : U \to \mathbb{R}^n$ $(n > m)$ is C^1, then $\vec{\phi}(X)$ has content zero.

So all manifolds in $\mathbb{R}^n$ (of dim. $< n$) have content zero; the same goes for graphs of C^0 functions over compact sets.

One more useful statement is: if ϕ_1, ϕ_2 are continuous functions on $D \subset \mathbb{R}^{n-1}$,

& $S = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in D \;\middle|\; \phi_1\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \le x_n \le \phi_2\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\}$,

then $\displaystyle\int_S f\, dV_n = \int_D \left(\int_{\phi_1\left(\begin{smallmatrix} x_1 \\ \vdots \\ x_{n-1} \end{smallmatrix}\right)}^{\phi_2\left(\begin{smallmatrix} x_1 \\ \vdots \\ x_{n-1} \end{smallmatrix}\right)} f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} dx_n \right) dV_{n-1}$.



I should also point out that the definition of n-volume of $S \subset \mathbb{R}^n$ is given by

$$\mathrm{vol}(S) := \int_S 1\, dV.$$

Ex/ $$\mathrm{vol}\big(\underbrace{[a_1,b_1]\times\cdots\times[a_n,b_n]}_{R}\big) = \int_R 1\,dV$$

$$= \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} 1\,dx_n\cdots dx_1$$

$$= \left(\int_{a_1}^{b_1} 1\,dx_1\right)\cdots\left(\int_{a_n}^{b_n} 1\,dx_n\right) = \prod_{i=1}^{n}(b_i-a_i).$$

//

Ex/ $$\int_{[0,1]^n} \frac{1}{1-x_1\cdots x_n}\,dV \overset{\text{Fubini}}{=}$$

$$\int_0^1\cdots\left(\int_0^1 \frac{1}{1-x_1\cdots x_n}\,dx_1\right)\cdots dx_n =$$

$$\int_0^1\cdots\left(\int_0^1 \sum_{k=0}^{\infty} x_1^k\cdots x_n^k\,dx_1\right)\cdots dx_n =$$

Uniform convergence doesn't quite hold, and an argument is required (will skip it)

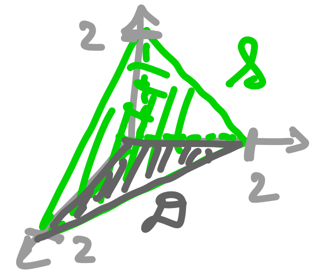
$$\sum_{k=0}^{\infty}\int_0^1\cdots\left(\int_0^1 x_1^k\cdots x_n^k\,dx_1\right)\cdots dx_n =$$

$$\sum_{k=0}^{\infty}\left(\int_0^1 x^k\,dx\right)^n = \sum_{k=0}^{\infty}\frac{1}{(k+1)^n} = 1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots$$

$$=: \zeta(n)$$

//

PROBLEM: Calculate the triple integral $\int_S y\,dV$, where S is the tetrahedron bounded by $x=0$, $y=0$, $z=0$, and $x+y+z=2$:



$$\iiint_S y\,dV = \iint_D\left(\int_0^{2-x-y} y\,dz\right)dA$$

$$= \iint_D y(2-x-y)\,dA$$

$$= \int_0^2\int_0^{2-x}(2y-xy-y^2)\,dy\,dx$$

$$= \int_0^2 \left[y^2-\tfrac{1}{2}xy^2-\tfrac{1}{3}y^3\right]_{y=0}^{2-x}dx$$

$$= \int_0^2\left(\frac{4}{3}-2x+x^2-\frac{1}{6}x^3\right)dx$$

$$= \left[\frac{4}{3}x - x^2+\frac{x^3}{3}-\frac{x^4}{24}\right]_0^2$$

$$= \frac{2}{3}.$$

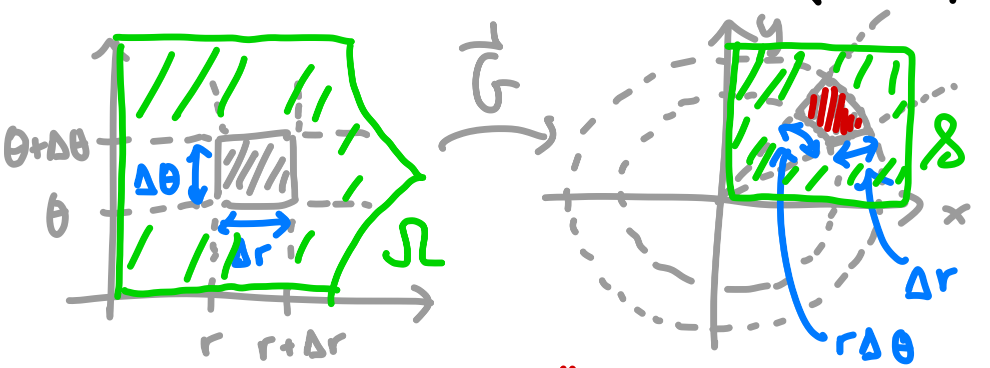
§3. Polar Integration

Consider the map

xy-plane

$$\vec{G} : [0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$$

sending $\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \vec{G}\begin{pmatrix} r \\ \theta \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$.



"$\Delta A = r\,\Delta r\,\Delta\theta$"

$\vec{G}(\Omega) = S$

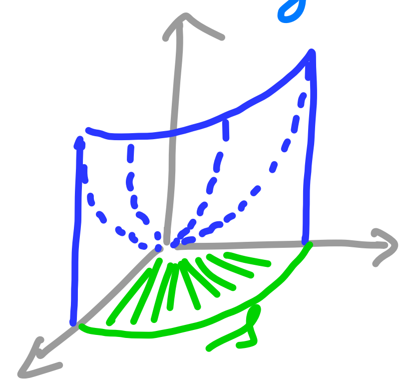
The picture suggests that if we use angular sector partitions of S, we should arrive at the alternate formula

$$\int_S f\begin{pmatrix} x \\ y \end{pmatrix} dA = \iint_\Omega f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} r\,dr\,d\theta$$

$$\left(= \int_\Omega (f\circ\vec{G})\begin{pmatrix} r \\ \theta \end{pmatrix} r\,dA \right)$$

This will be better justified once we have the change-of-variable formula.

Ex/ Find the volume of the solid in the 1st octant bounded by $z = x^2+y^2$, $x^2+y^2=4$, and the coordinate planes.



$$V = \iiint 1\,dV$$

$$= \iint_S (x^2+y^2)\,dA$$

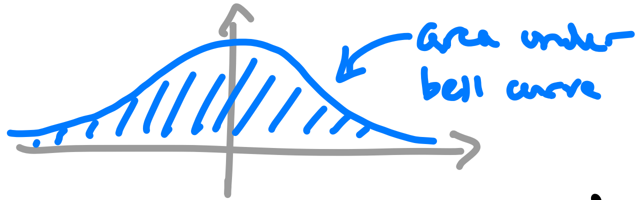
$$= \iint_\Omega \left(\overbrace{(r\cos\theta)^2 + (r\sin\theta)^2}^{r^2}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\int_0^2 r^3\,dr\right)d\theta$$

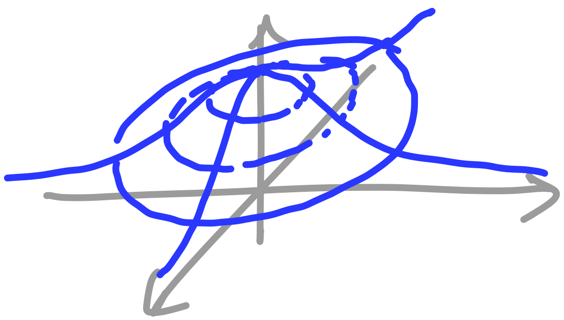
$$= \int_0^{\pi/2} \frac{2^4}{4}\,d\theta$$

$$= 4\cdot\frac{\pi}{2} = 2\pi. \quad //$$

Ex/ Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$.



Consider the volume V under $e^{-x^2-y^2}$



which we compute in two ways:

① $$V = \lim_{b\to\infty} \int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dx\,dy$$

$$= \lim_{b\to\infty} \left(\int_{-b}^{b} e^{-x^2}\,dx\right)^2$$

$$= \left(\lim_{b\to\infty} \int_{-b}^{b} e^{-x^2}\,dx\right)^2$$

$$= I^2.$$

② $$V = \lim_{a\to\infty} \int_0^{2\pi}\int_0^{a} e^{-r^2}\,r\,dr\,d\theta$$

$$= \lim_{a\to\infty} \int_0^{2\pi} \left[-\tfrac{1}{2}e^{-r^2}\right]_0^a d\theta$$

$$= \lim_{a\to\infty} \tfrac{1}{2}\int_0^{2\pi} \left(1 - e^{-a^2}\right) d\theta$$

$$= \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right)$$

$$= \pi.$$

So $I^2 = \pi \implies$

$$I = \sqrt{\pi}.$$

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