Multiple Integrals
Last time we defined a bounded function \( f \) on a rectangle \( R \subset \mathbb{R}^2 \) to be \underline{integrable} if we can make
\[
\int_R \tilde{f} \, dA - \int_R f \, dA \approx 0
\]
arbitrarily small, where \( 0 \leq f \leq \tilde{f} \) are step functions. In that case we defined
\[
\int_R f \, dA := \sup \left\{ \int_R g \, dA \mid 0 \leq g \leq f \right\} = \inf \left\{ \int_R g \, dA \mid \tilde{f} \leq g \right\}.
\]

We showed that all bounded functions whose discontinuity set \( D \subset R \) has content zero (e.g., piecewise smooth curves) are integrable. This opened the door to defining, for \( f \) a continuous function on a region \( R \subset \mathbb{R}^2 \),
\[
\int_R f \, dA := \int_R \tilde{f} \, dA
\]
where \( \tilde{f} \) is the (integrable!) extension by-zero of \( f \) to all of \( R \).

Integrals over \( R \) can be computed by \underline{Fubini's Theorem}: If \( f: R \to \mathbb{R} \) is integrable on \( R \), \( f(y_0) \) is integrable on \([a,b]\), for each \( y_0 \in [c,d] \), and \( A(y) := \int_a^b f(x) \, dx \) is integrable on \([c,d]\), then
\[
\int_R f \, dA = \int_c^d A(y) \, dy.
\]
Specifically, we had that sets of

**Type I:** \( \{ (y) \mid a \leq x \leq b, \; y_i(x) \leq y \leq y_2(x) \} \)

and

**Type II:** \( \{ (x) \mid c \leq y \leq d, \; \psi(y) \leq x \leq \phi(y) \} \)

(with \( \phi, \psi \) continuous) are regions.

So continuous functions on them are integrable, and we may use Fubini to compute their integrals: given

\[ f : \mathbb{R} \to \mathbb{R} \text{ C}^0, \]

extend \( f \) by 0 on \( \mathbb{R} \setminus \mathbb{R} \) to get \( \tilde{f} \).

Then

\[
\int_{\mathbb{R}} f \, dA := \int_{\mathbb{R}} \tilde{f} \, dA
\]

where \( \tilde{f} \) is the (integrable!) extension-
by-zero of \( f \) to all of \( \mathbb{R} \).

Integrals over \( \mathbb{R} \) can be computed by

**FUBINI’S THEOREM:** If \( f : \mathbb{R} \to \mathbb{R} \) is
integrable on \( \mathbb{R} \), \( f(y_0) \) is integrable on \([a, b] \) for each \( y_0 \in [c, d] \), and

\( A(y) := \int_{a}^{b} f(y) \, dx \) is integrable on \([c, d] \), then

\[
\int_{\mathbb{R}} f \, dA = \int_{c}^{d} A(y) \, dy.
\]

and when we left off we were starting to apply this to compute
integrals over nonrectangular regions of “type I” and “type II”.

\[
\int_{\mathbb{R}} f \, dA = \int_{\mathbb{R}} \tilde{f} \, dA
\]
Specifically, we had two sets of types:

**Type I:** \( \{(y) \mid a \leq x \leq b, \ \psi_1(y) \leq y \leq \psi_2(y)\} \)

**Type II:** \( \{(x) \mid c \leq x \leq d, \ \gamma_1(y) \leq x \leq \gamma_2(y)\} \)

(with \( \psi_i, \gamma_i \) continuous) are regions.

So continuous functions on them are integrable, and we may use Fubini to compute their integrals: given \( f : \mathbb{R} \to \mathbb{R} \) is \( C^0 \), extend \( f \) by \( 0 \) on \( \mathbb{R} \setminus B \) to get \( \tilde{f} \); then

\[
\iint_B f \, dA = \iint_B \tilde{f} \, dA
\]

By Fubini,

\[
\iint_B \tilde{f} \, dA = \int_c^d \left( \int_a^b \tilde{f}(y) \, dx \right) \, dy
\]

\[
= \int_c^d \left( \int_a^b f(y) \, dx \right) \, dy
\]

\[
= \int_c^d \left( \int_a^b \psi_1(y) \, f(y) \, dx \right) \, dy
\]

**Example:**

Compute \( \int_B (y^2 - x) \, dA \)

\[
= \int_0^1 \left( \int_0^{y^2} (y^2 - x) \, dx \right) \, dy
\]

\[
= \int_0^1 \left[ x(y^2) \right]_{x=0}^{x=y^2} \, dy
\]

\[
= \int_0^1 \left[ y^2 \right]_{x=0}^{x=y^2} \, dy
\]

\[
= \int_0^1 (y^2 - y) (y+1) \, dy
\]

\[
= -\frac{65}{6}
\]
Problem: Find $\int_S (8x + 10y) \, dA$, where $S$ is the region between the graphs of $y = x^2$ and $y = 2x$.

\[
\int_0^2 \left( \int_{x^2}^{2x} (8x + 10y) \, dy \right) \, dx
\]

\[
= \int_0^2 [8xy + 5y^2]_{y=x^2}^{2x} \, dx
\]

\[
= \int_0^2 [(16x^2 + 20x^2) - (8x^3 + 5x^4)] \, dx
\]

\[
= \int_0^2 (-5x^4 - 8x^3 + 3x^2) \, dx
\]

\[
= ... = 32.
\]

What if $S$ is more complicated?

(That is, neither type I nor type II.)
\[ \int_3 1 \, dA = \int_{\Delta_1} 1 \, dA + \int_{\Delta_2} 1 \, dA \]

(by the way, this computes one of \( \beta \))

\[ = \int_0^1 \int_{x^2/8}^{x^2} 1 \, dy \, dx + \int_{x^2/8}^{\sqrt{x}} 1 \, dy \, dx \]

\[ = \int_0^1 \left( x^2 - \frac{x^2}{8} \right) \, dx + \int_{x^2/8}^{\sqrt{x}} \left( \frac{1}{x} - \frac{x^2}{8} \right) \, dx \]

\[ = \ldots \]

\[ = \log(2). \]

What if \( \beta \) is more complicated?

\[ \beta = \beta_1 \cup \beta_\Pi \]

(and \( \beta_1 \cap \beta_\Pi \) is a segment)

(That is, neither type I nor type II.)

Split it into subsets and write

\[ \int_\beta 1 \, dA = \int_{\beta_1} 1 \, dA + \int_{\beta_\Pi} 1 \, dA. \]

More generally, you can use this technique (of breaking the region) to compute integrals which are of type I or II:

Ex: Find \( \int_\beta 1 \, dA \), where \( \beta \) is in the first quadrant, bounded by \( y=x^2 \), \( y=\frac{x^2}{8} \), and \( y=\frac{1}{x} \).
Ex. Find $\int_{[0,1] \times [0,1]} f(y) \, dA$, where

$$f(y) := \begin{cases} 1 & x=0 \text{ or } y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

First, $f$ is integrable on its discontinuity set is contained in a set of content zero, the line segment $D$ shown. We cannot use Fubini in the form $\int_0^1 \left( \int_0^1 f(y) \, dy \right) \, dx$, be $f(y)$ is non-integrable. However, we can use it in the form $\int_0^1 \left( \int_0^1 f(y) \, dx \right) \, dy$, and

$$\int_0^1 f(y) \, dx = 0 \text{ for each } y \text{ (why?)};$$

so $\int_{[0,1]^2} f \, dA = 0$.

Problem: Find $\int_0^1 \left( \int_{x/2}^{2y} e^y \, dy \right) \, dx = I$

Well, of course you can't antidifferentiate $e^{x^2}$. But you also can't just "exchange the integrals" — you'd end up with outer limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration (by using Fubini twice).

$$I = \int_0^1 e^{y^2} \, dA = \int_0^1 \left( \int_0^{y^2} e^y \, dx \right) \, dy$$

$$= \int_0^1 2y e^{y^2} \, dy = \left[ e^{y^2} \right]_0^1$$

$$= e^1 - 1.$$
If moreover $\frac{\partial F}{\partial x}$ is assumed $C^0$, then
\[ \phi(t) := \int_c^t \frac{\partial F}{\partial x}(y) \, dy \]
is defined, and by Fubini:
\[
\int_a^x \phi(t) \, dt = \int_a^x \int_c^t \frac{\partial F}{\partial x}(y) \, dy \, dt \\
= \int_c^x \int_a^t \frac{\partial F}{\partial x}(y) \, dt \, dy \\
= \int_c^x \left( f(x) - f(y) \right) \, dy \\
= F(x) - F(a)
\]

\[ \Rightarrow \quad F'(x) = \frac{d}{dx} \int_a^x \phi(t) \, dt = \phi(x) \]

\[ \Rightarrow \quad \frac{d}{dx} \int_c^d f(y) \, dy = \int_c^d \frac{\partial f}{\partial x}(y) \, dy, \]

that is, we can "differentiate under the integral sign."

Finally, a bit of theory: if $f \leq g$ on $\delta$, then $f \leq \tilde{g}$ on $R$ hence the superfunction $0 \leq \tilde{g} - \tilde{f}$ on $R$ implies
\[
0 \leq \int_R (\tilde{g} - \tilde{f}) \, dA = \int_R \tilde{g} \, dA - \int_R \tilde{f} \, dA \\
\Rightarrow \int_R f \, dA \leq \int_R g \, dA. \quad \text{Hence}
\]

\[ -|f| \leq f \leq |f| \quad \Rightarrow \quad \int |f| \, dA \leq \int f \, dA \leq \int |f| \, dA
\]

\[ \Rightarrow \quad \left| \int_R f \, dA \right| \leq \int_R |f| \, dA \quad \text{in general.}
\]

Now suppose $f(x)$ is a continuous (hence uniformly so) function on $R = [a,b] \times [c,d]$. Pick $\delta > 0$ so that
\[ ||f(x) - f(x_0)|| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{\varepsilon}{c-d}. \]

Then $F(x) := \int_c^d f(x) \, dy$ is also $C^0$ since
\[ |F(x) - F(x_0)| = \left| \int_c^d f(x) - f(x_0) \, dy \right| \leq \int_c^d |f(x) - f(x_0)| \, dy < \varepsilon. \]
Chapter 2. Multiple Integrals

These work just like double integrals: let \( S \subset \mathbb{R}^n \) be a bounded set, \( f : S \to \mathbb{R} \) a function on it, and \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) its extension by 0 to some enclosing box \( R = [a_1, b_1] \times \cdots \times [a_n, b_n] \). Subdividing \( R \) into \( R_i = (i_1, \ldots, i_n) \) we define step functions by \( s_i |_{R_i} \equiv c_i \), their integrals by \( \int_{R_i} s_i \, dV := \sum_i c_i \cdot \text{vol}(R_i) \), and define

\[
\text{if integrable} \quad \iff \quad \tilde{f} \text{ integrable}
\]

(on \( S \)) \quad (on \( R \))

\[
\text{def.} \quad \begin{cases}
\sup \{ \int_S s \, dV \mid s \leq f \text{ step} \} \\
\inf \{ \int_R \tilde{f} \, dV \mid \tilde{f} \geq f \text{ step} \}
\end{cases}
\]

In which case this common value defines \( \int_S f \, dV \).
Theorem 1: If \( \mathbb{R}^n \) has content zero (can be covered by boxes of arbitrary small total volume), and \( f \) is \( C^0 \), then \( f \) is integrable.

Theorem 2 (Fubini): If everything is integrable,

\[
\int_{\mathbb{R}^n} f \, \mathrm{d}V = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} \left( \int_{a_3}^{b_3} \cdots \left( \int_{a_n}^{b_n} f(x_1, \ldots, x_n) \, dx_n \right) \cdots \right) \, dx_2 \right) \cdots \, dx_1
\]

(or in any other order).

The proofs are the same. We also have

Theorem 3: If \( \mathcal{X} \subset \mathbb{R}^m \) is compact, \( U \subset \mathbb{R}^m \) is open and contains \( \mathcal{X} \), and \( \varphi : U \to \mathbb{R}^n \) (\( n > m \)) is \( C^1 \), then \( \varphi(\mathcal{X}) \) has content zero.

So all manifolds in \( \mathbb{R}^n \) (\( n \) dim. \( \leq n \)) have content zero; the same goes for graphs of \( C^0 \) functions over compact sets.

Proof of Theorem 3: we may assume \( \mathcal{X} \subset [0,1]^m \). Subdivide \( [0,1]^m \) into \( \mathbb{N}^m \) cubes of volume \( (1/\mathbb{N})^m \), and let \( K \) be the union of cubes that meet \( \mathcal{X} \). By taking \( \mathbb{N} \) large, we may assume \( K \subset U \).

Since \( K \) is compact, \( D\varphi \) is uniformly continuous on \( K \); so \( \|D\varphi\| \leq M \) then. By the mean value inequality, for \( a, b \) in the same "little cube" we have

\[
\|\varphi(b) - \varphi(a)\| \leq M \|b - a\| \leq \frac{M\sqrt{m}}{N}.
\]

Pick any point in the image of this little cube and draw a cube centered over it with side length \( \frac{2M\sqrt{m}}{N} =: \frac{k}{N} \); it contains the image of the little cube \( \mathcal{B} \) has volume \( \frac{k^m}{N^m} \). Their union contains \( \varphi(\mathcal{X}) \) and has total volume \( \leq \frac{k^n}{N^n} \cdot \frac{k^m}{N^m} = \frac{k^{n+m}}{N^{n+m}} \to 0 \) as \( N \to \infty \). \( \square \)
Theorem 1: If \( \mathcal{D} \) has content zero (can be covered by boxes of arbitrarily small total volume), and \( f \) is \( C^0 \), then \( f \) is integrable.

Theorem 2 (Fubini): If everything is integrable,
\[
\int_{\mathcal{D}} f \, dV = \int_{a_n}^{b_n} \cdots \left( \int_{a_2}^{b_2} \left( \int_{a_1}^{b_1} f(x_1, \ldots, x_n) \, dx_n \right) \, dx_2 \right) \cdots \, dx_1
\]
(or in any other order).

The proofs are the same. We also have

Theorem 3: If \( \mathcal{S} \subset \mathbb{R}^n \) is compact, \( \mathcal{U} \subset \mathbb{R}^m \) is open and contains \( \overline{\mathcal{S}} \), and \( \phi : \mathcal{U} \to \mathbb{R}^n \) (\( n > m \)) is \( C^1 \), then \( \phi(\overline{\mathcal{S}}) \) has content zero.

So all manifolds in \( \mathbb{R}^n \) (\( n \leq \text{dim.} \mathcal{U} \)) have content zero; the same goes for graphs of \( C^0 \) functions over compact sets.

One more useful statement is: if \( \phi_1, \phi_2 \) are continuous functions on \( \mathcal{D} \subset \mathbb{R}^{n-1} \),
\[
\mathcal{S} = \left\{ (x_n) \in \mathcal{D} \mid \phi_1(x_n) \leq x_n \leq \phi_2(x_n) \right\},
\]
then
\[
\int_{\mathcal{D}} f \, dV = \int_{\mathcal{D}} \left( \int_{\phi_1(x_n)}^{\phi_2(x_n)} f(x_n) \, dx_n \right) \, dV_{n-1}.
\]

I should also point out that the definition of \( n \)-volume of \( \mathcal{S} \subset \mathbb{R}^n \) is given by
\[
\text{vol}(\mathcal{S}) := \int_{\mathcal{D}} 1 \, dV.
\]
Ex/ \quad \text{vol} \left( [a_1, b_1] \times \cdots \times [a_n, b_n] \right) = \int_{R^n} 1 \, dV \\
= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} 1 \, dx_n \cdots dx_1 \\
= \left( \int_{a_i}^{b_i} 1 \, dx_i \right) \cdots \left( \int_{a_n}^{b_n} 1 \, dx_n \right) = \prod_{i=1}^{n} (b_i - a_i).

Ex/ \quad \int_{[0,1]^n} \frac{1}{1 - x_1 \cdots x_n} \, dV = \prod_{i=1}^{n} \int_{0}^{1} \frac{1}{1 - x_i} \, dx_i \\
\downarrow \quad \text{Uniform convergence doesn't quite hold, and an argument is required (will skip it)} \\
\sum_{k=0}^{\infty} \left( \int_{0}^{1} x^k \, dx \right)^n = \sum_{k=0}^{\infty} \frac{1}{(kn)^n} = 1 + \frac{1}{2n} + \frac{1}{3n} + \cdots \\
= : \zeta(n).

\underline{PROBLEM:} Calculate the triple integral \\
\int_{\Omega} y \, dV, \quad \text{where } \Omega \text{ is the tetrahedron bounded by } x = 0, \ y = 0, \ z = 0, \ \text{and } x + y + z = 2. \\
\mathcal{M}_f y \, dV = \iint_{\partial \Omega} \left( \int_{0}^{2-x-y} y \, dx \right) \, dA \\
= \iint_{\partial \Omega} y (2 - x - y) \, dA \\
= \int_{0}^{2} \int_{0}^{2-x} (2y - x - y) \, dy \, dx \\
= \int_{0}^{2} \left( y^2 - \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \bigg|_{y=0}^{y=2-x} \, dx \\
= \int_{0}^{2} \left( \frac{y^4}{3} - 2x + x^3 - \frac{1}{6} x^3 \right) \, dx \\
= \left[ \frac{y^5}{5} - x^2 + \frac{x^4}{3} - \frac{x^4}{24} \right]_{0}^{2} \\
= \frac{2}{3}. 
2.3. Polar Integration

Consider the map
\[ G : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \]
\[ (r, \theta) \mapsto \mathbf{G}(\theta) = (r \cos \theta, r \sin \theta). \]

The picture suggests that if we use angular sector partitions of \( S \), we should arrive at the alternate formula
\[ \int_{S} f(x, y) \, dA = \int_{S} f(r \cos \theta, r \sin \theta) \, \cancel{r} \, dr \, d\theta \]
\[ = \int_{S} (f(r \cos \theta)(r) \, \cancel{r}) \, dr \, d\theta \]
\[ = \int_{0}^{\infty} \left( \int_{0}^{2\pi} f(r \cos \theta) \, d\theta \right) r \, dr. \]

This will be better justified once we have the change-of-variable formula.

Example: Find the volume of the solid in the 1st octant bounded by
\[ z = x^2 + y^2, \quad x^2 + y^2 = 4, \]
and the coordinate planes.

\[ V = \iiint 1 \, dV \]
\[ = \iiint_S (x^2 + y^2) \, r \, dr \, d\theta \]
\[ = \iint_S \left( (r \cos \theta)^2 + (r \sin \theta)^2 \right) \, r \, dr \, d\theta \]
\[ = \sqrt{2} \left( \int_{0}^{\pi/2} (r^3) \, dr \right) \, d\theta \]
\[ = \frac{\pi}{2} \frac{4}{3} \, d\theta \]
\[ = 4 \cdot \frac{\pi}{2} = 2\pi. \]
Ex: Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

Consider the volume \( V \) under \( e^{-x^2-y^2} \)

which we compute in two ways:

1. \( V = \lim_{b \to \infty} \int_{-b}^{b} \int_{-b}^{b} e^{-x^2-y^2} \, dx \, dy \)
   \[ = \lim_{b \to \infty} \left( \int_{-b}^{b} e^{-x^2} \, dx \right)^2 \]
   \[ = \left( \lim_{b \to \infty} \int_{-b}^{b} e^{-x^2} \, dx \right)^2 \]
   \[ = I^2 \]

2. \( V = \lim_{a \to \infty} \int_{0}^{2\pi} \int_{0}^{a} e^{-r^2} \, r \, dr \, d\theta \)
   \[ = \lim_{a \to \infty} \int_{0}^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \right]_0^a \, d\theta \]
   \[ = \lim_{a \to \infty} \frac{1}{2} \int_{0}^{2\pi} (1 - e^{-a^2}) \, d\theta \]
   \[ = \lim_{a \to \infty} \pi (1 - e^{-a^2}) \]
   \[ = \pi. \]

So \( I^2 = \pi \implies I = \sqrt{\pi} \).