Integrals and Determinants

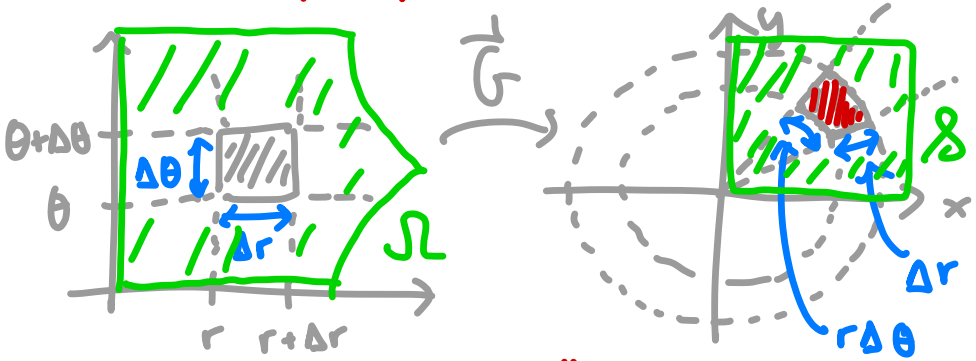
19

§1. Polar, cylindrical, spherical

POLAR COORDINATES

$\vec{G}: [0,\infty) \times [0, 2\pi) \longrightarrow \mathbb{R} \times \mathbb{R}$

$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$



$\vec{G}(\Omega) = S$

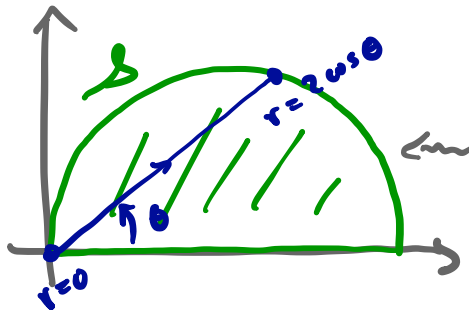
"$\Delta A = r\Delta r \Delta\theta$"

$$\int_S f(x,y)\,dA = \int_\Omega f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex/ Evaluate the iterated integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx .$$

This is nasty to antidifferentiate in x or y, so we convert to polar:



$y = \sqrt{2x-x^2}$

$y^2 = 2x - x^2$

$(x-1)^2 + y^2 = 1$

$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

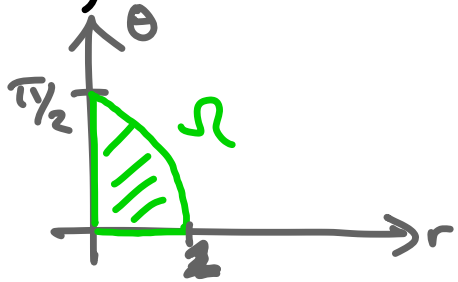
$$\iint_S r\,dA = \int_0^{\pi/2}\left(\int_0^{2\cos\theta} r^2\,dr\right)d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}(\cos\theta - \sin^2\theta\cos\theta)\,d\theta$$

$\cos\theta(1-\sin^2\theta)$

$$= \frac{8}{3}\left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \frac{8}{3}\left(1-\frac{1}{3}\right) = \frac{16}{9}. \quad //$$

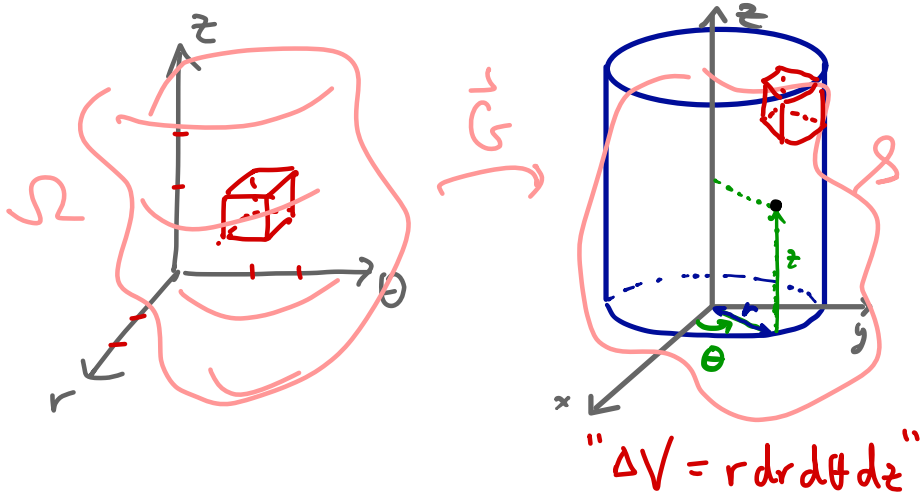
To connect more explicitly to this formula, here $f\begin{pmatrix}x\\y\end{pmatrix} = \sqrt{x^2+y^2} \Rightarrow f\begin{pmatrix}r\cos\theta\\r\sin\theta\end{pmatrix} = r$ and Ω looks like



§1. Polar, cylindrical, spherical

CYLINDRICAL COORDINATES

$$\vec{G} : [0,\infty) \times [0, 2\pi) \times \mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}\times\mathbb{R}$$
$$(r,\ \theta,\ z) \longmapsto (r\cos\theta, r\sin\theta, z)$$

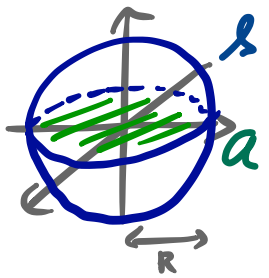


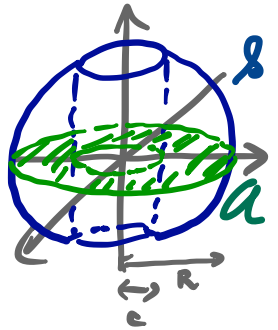
"$\Delta V = r\,dr\,d\theta\,dz$"

$$\int_S f\begin{pmatrix}x\\y\\z\end{pmatrix} dV = \int_\Omega f\begin{pmatrix}r\cos\theta\\r\sin\theta\\z\end{pmatrix} r\,dr\,d\theta\,dz$$

Ex/ Compute the volume of a sphere of radius R.

Also find the volume of a cored apple, if the apple is a sphere of radius R and the core is its intersection with a solid cylinder of radius c.





$$V = \int_S 1\,dV = \int_S 1\; r\,dr\,d\theta\,dz$$
$$= \int_c^R \left(\int_0^{2\pi} \left(\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} 1\,dz \right) d\theta \right) r\,dr$$
$$= 4\pi \int_c^R \sqrt{R^2-r^2}\; r\,dr$$
$$= 4\pi \left[-\tfrac{1}{3}(R^2-r^2)^{3/2} \right]_c^R = \tfrac{4}{3}\pi (R^2-c^2)^{3/2}$$
$$= \tfrac{4}{3}\pi h^3, \quad h = \text{height of cored apple}$$

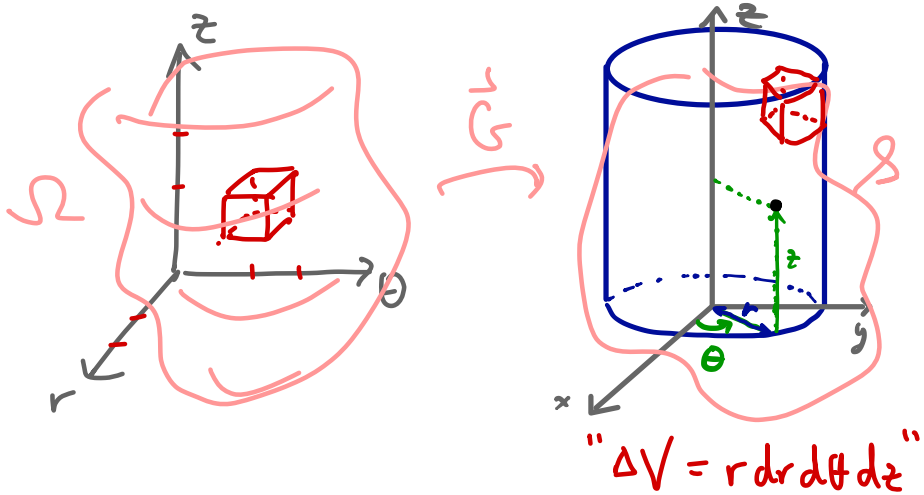
($h = R$ if not cored) //

§1. Polar, cylindrical, spherical

CYLINDRICAL COORDINATES

$$\vec{G}: [0,\infty)\times[0,2\pi)\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}\times\mathbb{R}$$

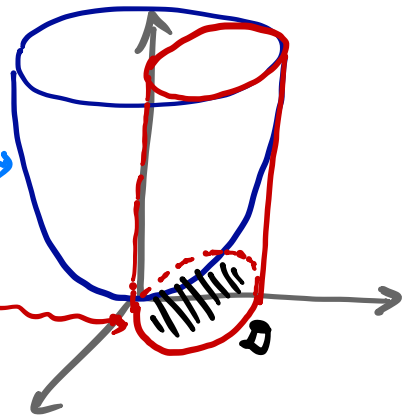
$$(r,\ \theta,\ z) \longmapsto (r\cos\theta, r\sin\theta, z)$$



"$\Delta V = r\,dr\,d\theta\,dz$"

$$\int_S f\begin{pmatrix}x\\y\\z\end{pmatrix} dV = \int_\Omega f\begin{pmatrix}r\cos\theta\\ r\sin\theta\\ z\end{pmatrix} r\,dr\,d\theta\,dz$$

PROBLEM: Calculate $\int_S z^{-1/2}\,dV$, when S is the region bounded by $z=0$, $z=x^2+y^2$, & $x^2+y^2=2y$.



$$\int_S z^{-1/2}\,dV = \iint_D \left(\int_0^{x^2+y^2} z^{-1/2}\,dz\right) dx\,dy$$

$$= \int_0^{\pi}\left(\int_0^{2\sin\theta}\left(\int_0^{r^2} z^{-1/2}\,dz\right) r\,dr\right) d\theta$$

$$= \int_0^{\pi}\left(\int_0^{2\sin\theta}\left(2z^{1/2}\Big|_0^{r^2}\right) r\,dr\right) d\theta$$

$$= \int_0^{\pi}\left(\int_0^{2\sin\theta} 2r^2\,dr\right) d\theta$$

$$= \int_0^{\pi} \frac{16}{3}\sin^3\theta\,d\theta = \frac{16}{3}\int_0^{\pi}(\sin\theta - \sin\theta\cos^2\theta)\,d\theta$$

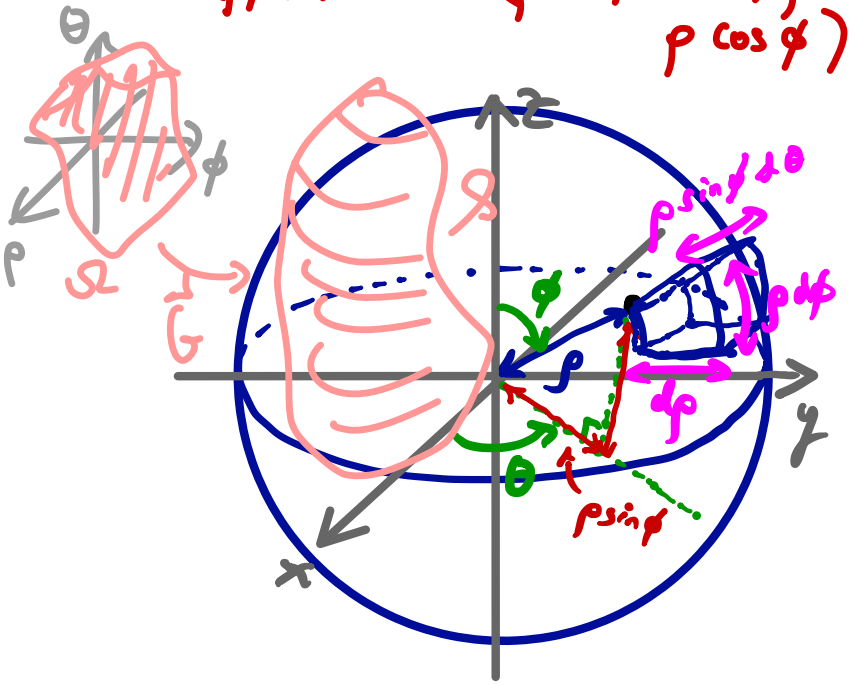
$$= \frac{16}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = 64/9\ .$$

§1. Polar, cylindrical, spherical

SPHERICAL COORDINATES

$$\vec{G} : [0,\infty) \times [0,\pi] \times [0,2\pi) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

$$(\rho, \phi, \theta) \mapsto (\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)$$



"$\Delta A = (\Delta\rho)(\rho\,\Delta\phi)(\rho \sin\phi\,\Delta\theta)$"

$$\int_S f\begin{pmatrix} x \\ y \\ z \end{pmatrix} dV = \int_\Omega f\begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Ex/ This makes finding the volume of a 3-ball (solid sphere) of radius R particularly easy:

$$\int_S 1\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

↑ the 3-ball

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{R} \rho^2\, d\rho\right)$$

$$= 2\pi \cdot \left[-\cos\phi\right]_0^{\pi} \cdot \left[\frac{\rho^3}{3}\right]_0^{R}$$

$$= 2\pi \cdot 2 \cdot \frac{R^3}{3}$$

$$= \frac{4}{3}\pi R^3. \quad //$$

§2. Masses and moments

CENTER OF MASS

Lamina := flat sheet so thin we can think of it as 2-D; made of nonhomogeneous material (density can vary)

If a lamina covers a region $S \subset \mathbb{R}^2$

with mass density function $\delta : S \to \mathbb{R}$, we may extend δ to Q by defining it to be zero off S.

Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}$.

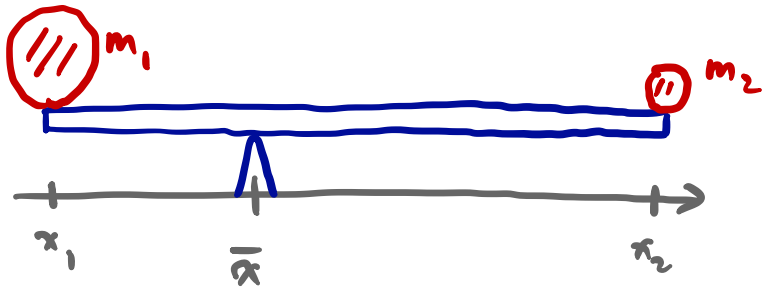
Then the mass $m(Q_{ij}) \approx \delta(q_{ij})\, a(Q_{ij})$

$\Rightarrow$ total mass $m(S) = m(Q) \approx \sum_{i,j} \delta(q_{ij})\, a(Q_{ij})$.

By integrability of $\delta(x,y)$, taking the size of the partition to 0 makes the integrals of the upper & lower step functions coincide, so

$$m = \iint_S \delta \, dA \,. \qquad (\text{mass of lamina})$$

Now let's think about seesaws:



for this to balance, we need

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}). \qquad \text{equality of moments}$$

Solving for $\bar{x}$ gives $(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$

$$\Rightarrow \quad \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \,.$$

For n masses, one argues by induction that the balance point would be

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \qquad \left(\text{if in 2-D, } \bar{y} = \frac{\sum y_i m_i}{\sum m_i} \right).$$

Thinking of the lamina as a bunch of small masses (at q_{ij}) yields

$$\bar{x} \approx \frac{\sum_i \sum_j x_{ij}\, \delta(q_{ij})\, a(Q_{ij})}{\sum_i \sum_j \delta(q_{ij})\, a(Q_{ij})} \qquad \text{etc.}$$

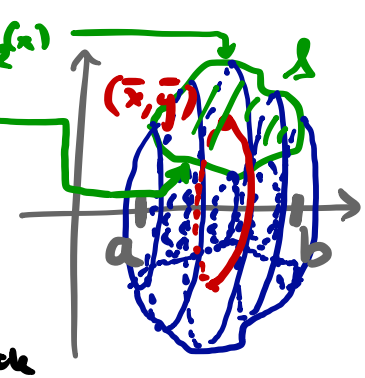
Hence (taking the limit)

$$\bar{x} = \frac{\iint_{\mathcal{S}} x\, \delta(x,y)\, dA}{\iint_{\mathcal{S}} \delta(x,y)\, dA} = \frac{M_y}{m}, \qquad \bar{y} = \frac{\iint_{\mathcal{S}} y\, \delta(x,y)\, dA}{\iint_{\mathcal{S}} \delta(x,y)\, dA} = \frac{M_x}{m}.$$

where M_x & M_y are the moments about the x- & y-axes, and $(\bar{x}, \bar{y})$ is the center of mass.

In the case where δ is constant, this is called the centroid of $\mathcal{S}$.

PAPPUS'S THEOREM



Suppose we now rotate $\mathcal{S}$ about the x-axis, so that the centroid traces out a circle of radius $\bar{y}$. The resulting solid has volume

$$V = \int_a^b \pi \left(\phi_2(x)^2 - \phi_1(x)^2 \right) dx$$

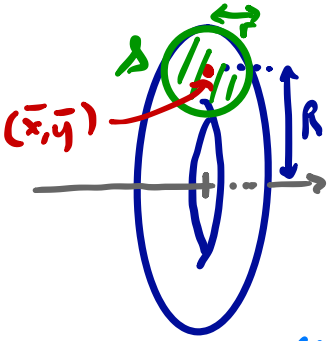
$$= \pi \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 2y\, dy \right) dx$$

$$= 2\pi \iint_{\mathcal{S}} y\, dA = 2\pi M_x$$

$$= 2\pi \bar{y} \iint_{\mathcal{S}} dA = 2\pi \bar{y}\, a(\mathcal{S})$$

where the mass density $\delta \equiv 1$.

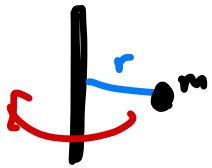
Ex/ Find the volume of the torus T depicted:



$$v(T) = 2\pi \bar{y}\, a(\mathcal{S})$$

$$= 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R. \quad //$$

MOMENT OF INERTIA

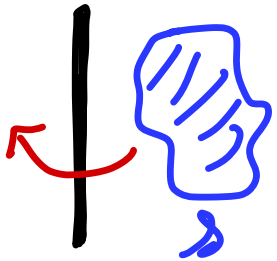


The kinetic energy of a point mass m rotating about an axis with angular speed $\omega = \frac{v}{r} = \frac{\text{speed}}{\text{radius}}$ is $\frac{1}{2}mv^2 = \frac{1}{2}\underbrace{(mr^2)}_{I}\omega^2$.

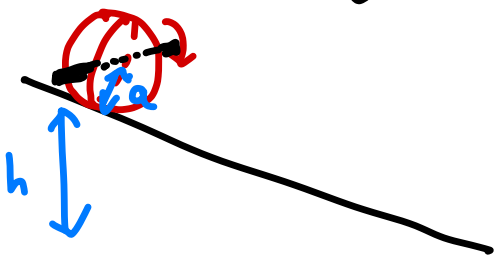
The moment of inertia I generalizes to continuous mass distributions; by the same argument as for center of mass,

$$I = \int_S \delta r^2 \, dV .$$

(δ: mass density; dV or dA)



Consider an object rolling down an incline with "rolling radius" a:



original potential energy = final kinetic energy (translational + rotational)

$$\Downarrow$$

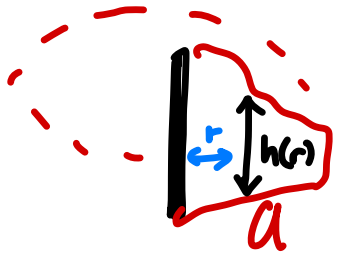
$$mgh = \frac{1}{2}ms^2 + \frac{1}{2}I\left(\frac{s}{a}\right)^2$$

(s: final linear speed; $\frac{s}{a}$: angular speed)

$$= \frac{1}{2}m\left(1 + \frac{I}{ma^2}\right)s^2$$

$\Rightarrow$ the smaller $\frac{I}{m}$ is, the larger s is and the faster the object reaches the bottom.

For a rotationally symmetric solid S of density $\delta = 1$ & "height" function $h(r)$:

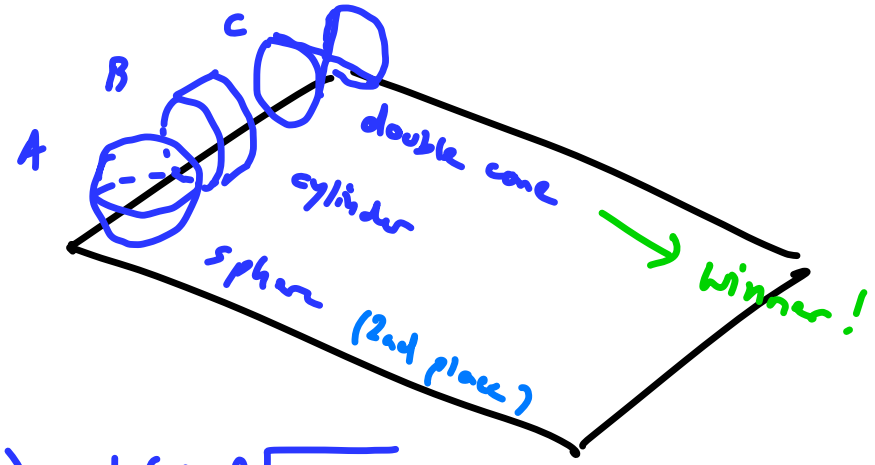


S = solid of revolution of a

If (say) $a = 1$ then we can use

$$\frac{I}{m} = \frac{\int_0^1 h(r)\, r^3 dr}{\int_0^1 h(r)\, r\, dr}$$

to decide which solid of revolution wins the race:



(A): $h(r) = 2\sqrt{1-r^2}$

(a) $\quad \frac{I}{m} = \frac{8\pi/15}{4\pi/3} = \frac{2}{5}$

(B): $h(r) = h$ (const.)

(a) $\quad \frac{I}{m} = \frac{\pi h/2}{\pi h} = \frac{1}{2}$ largest

(C): $h(r) = 2 - 2r$

(a) $\quad \frac{I}{m} = \frac{\pi/5}{2\pi/3} = \frac{3}{10}$ Smallest

original potential energy = final kinetic energy (translational + rotational)

$$\Downarrow$$

$$mgh = \frac{1}{2} m s^2 + \frac{1}{2} I \left(\frac{s}{a}\right)^2$$

(s: final linear speed; $\frac{s}{a}$: angular speed)

$$= \frac{1}{2} m \left(1 + \frac{I}{ma^2}\right) s^2$$

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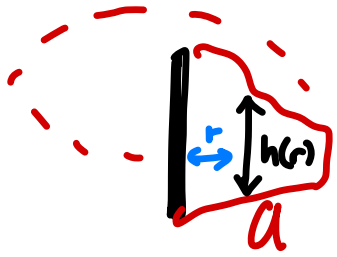

S = solid of revolution of a

we have (via cylindrical coordinates)

$$m = 2\pi \int_0^a h(r)\, r\, dr \quad \text{mass of } S$$

$$I = 2\pi \int_0^a h(r)\, r^3\, dr \quad \text{moment of inertia}$$

(ii) Antisymmetry:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_n \end{pmatrix} = -\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

for any 2 rows i & j

(iii) Normalization:

$$\det\begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix} = 1$$

i.e. the identity matrix.

§3. Determinants

Suppose you are looking for a function

$$\det : \left\{ \begin{matrix} n\times n \text{ real} \\ \text{matrices} \end{matrix} \right\} \to \mathbb{R}$$

with the following 3 properties, where we think of an $n\times n$ matrix A as an ordered n-tuple of row vectors

$$\begin{pmatrix} \leftarrow \vec{A}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{A}_n \rightarrow \end{pmatrix}:$$

(i) Multilinearity:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ a\vec{A}_i + b\vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} = a\cdot\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} + b\cdot\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

for any row i & constants $a, b \in \mathbb{R}$

Suppose you are looking for a function

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for any row i & constants $a, b \in \mathbb{R}$

(ii) Antisymmetry:

$$\det \begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_n \end{pmatrix} = - \det \begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

for any 2 rows i & j

(iii) Normalization:

$$\det \begin{pmatrix} +\vec{e}_1 \\ \vdots \\ +\vec{e}_n \end{pmatrix} = 1$$

i.e. the identity matrix.

MAIN THEOREM: For each n, such a function exists, and is unique!

Anticipating this result, we'll call this function the determinant, written $\det(A)$ or $|A|$.

Theorem 1: If 2 rows of A are equal, then $\det(A) = 0$.

Proof: If $\vec{A}_i = \vec{A}_j$, then by (ii),

$$\det(A) = -\det(A) \implies 2\det(A) = 0$$

$$\implies \det(A) = 0. \quad \square$$

Case 1: $a = c = 0$.

$$|A| = \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = \begin{vmatrix} b\vec{E}_2 \\ d\vec{E}_2 \end{vmatrix} \overset{(i)}{=} bd \begin{vmatrix} \vec{E}_2 \\ \vec{E}_2 \end{vmatrix} \overset{\text{Thm. 1}}{=} 0$$

Case 1': $a = 0$. (contains case 1)

$$|A| = \begin{vmatrix} b\vec{E}_2 \\ c\vec{E}_1 + d\vec{E}_2 \end{vmatrix} \overset{(i)}{=} bc \begin{vmatrix} \vec{E}_2 \\ \vec{E}_1 \end{vmatrix} + bd \begin{vmatrix} \vec{E}_2 \\ \vec{E}_2 \end{vmatrix}$$

$$= -bc \begin{vmatrix} \vec{E}_1 \\ \vec{E}_2 \end{vmatrix} \overset{(ii)}{=} -bc \overset{(iii)}{=} -bc.$$

Case 2: $a \neq 0$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \frac{c}{a} \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0 \text{ by Thm. 1}}$$

$$\overset{(i)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ c & \frac{bc}{a} \end{vmatrix} \overset{(i)}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix}$$

$$\overset{(i)}{=} \begin{vmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d - \frac{bc}{a} \end{vmatrix}}_{0 \text{ by case 1}} = \begin{vmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{vmatrix}$$

$$= a\left(d - \frac{bc}{a}\right) \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{=1 \text{ by (iii)}} = ad - bc.$$

So we get $|A| = ad - bc$ in each case.

(i) **Multilinearity**:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ a\vec{A}_i + b\vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} = a \cdot \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} + b \cdot \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

for any row i & constants $a, b \in \mathbb{R}$

(ii) **Antisymmetry**:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_n \end{pmatrix} = -\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

(iii) **Normalization**:

$$\det\begin{pmatrix} \vec{E}_1 \\ \vdots \\ \vec{E}_n \end{pmatrix} = 1$$

Theorem 1: If 2 rows of A are equal, then $\det(A) = 0$.

2 × 2 MATRICES

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the defining properties (i)–(iii) to compute $\det(A)$. (I will write $\vec{E}_i$ instead of ${}^t\vec{e}_i$ for the standard basis row vectors.)

A list $\sigma = (\sigma_1, \dots, \sigma_n)$ of numbers in $\{1, \dots, n\}$ is called a **permutation** if each number occurs once, in which case the **sign**

$$\operatorname{sgn}(\sigma) := (-1)^{\{\#\text{ of swaps required to get from } (1,\dots,n) \text{ to } \sigma\}}$$

By (ii) + (iii),

$$(*)\quad \det\begin{pmatrix} \vec{E}_{\sigma_1} \\ \vdots \\ \vec{E}_{\sigma_n} \end{pmatrix} = \begin{cases} \operatorname{sgn}(\sigma), & \text{if } \sigma \text{ is a permutation} \\ 0, & \text{otherwise;} \end{cases}$$

and so

$$\det(A) = \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_n \end{pmatrix} = \det\begin{pmatrix} \sum_{\sigma_1=1}^n a_{1,\sigma_1}\vec{E}_{\sigma_1} \\ \vdots \\ \sum_{\sigma_n=1}^n a_{1,\sigma_n}\vec{E}_{\sigma_n} \end{pmatrix}$$

$$\overset{(i)}{=} \sum_{\sigma_1=1}^n \cdots \sum_{\sigma_n=1}^n a_{1,\sigma_1}\cdots a_{n,\sigma_n} \cdot \det\begin{pmatrix} \vec{E}_{\sigma_1} \\ \vdots \\ \vec{E}_{\sigma_n} \end{pmatrix}$$

$$\overset{(*)}{=} \sum_{\substack{\sigma \\ \text{permutation}}} \left(\prod_{i=1}^n a_{i,\sigma_i}\right) \operatorname{sgn}(\sigma).$$

This proves uniqueness.

Existence is now clear as well: it is straightforward to check that the formula just derived satisfies (i)–(iii). □

(i) Multilinearity:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ a\vec{A}_i + b\vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} = a \cdot \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix} + b \cdot \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{B}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

for any row i & constants $a, b \in \mathbb{R}$

(ii) Antisymmetry:

$$\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_n \end{pmatrix} = -\det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{pmatrix}$$

(iii) Normalization:

$$\det\begin{pmatrix} \vec{E}_1 \\ \vdots \\ \vec{E}_n \end{pmatrix} = 1$$

To conclude, I'll give a quick sketch of proof of the

MAIN THEOREM: For each n, a function $\det : \{n \times n \text{ matrices}\} \to \mathbb{R}$ satisfying (i)–(iii) exists, and is unique!

Sketch: Suppose we have such a function (to prove uniqueness first).