# DETERMINANTS

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Today we'll continue our brief study of determinants, the goal of which is to prepare for the change-of-variable formula in multiple integrals. At the end of the last lecture, we proved that there is (for each n) exactly one function

$$\det : \left\{ \begin{matrix} n \times n \\ \text{matrices} \end{matrix} \right\} \to \mathbb{R}$$

satisfying three properties:

① **Normalization**:

$$\det(I_n) = 1.$$

$I_n$: $n \times n$ identity matrix

② **Antisymmetry in the rows**:

If two rows are swapped and all the other rows stay the same, the determinant gets multiplied by $-1$.

③ **Multilinearity in the rows**:

$$\det \begin{pmatrix} \leftarrow \vec{A}_1 \rightarrow \\ \vdots \\ \leftarrow a\vec{A}_i + b\vec{B}_i \rightarrow \\ \vdots \\ \leftarrow \vec{A}_n \rightarrow \end{pmatrix} = a \det \begin{pmatrix} \leftarrow \vec{A}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{A}_i \rightarrow \\ \vdots \\ \leftarrow \vec{A}_n \rightarrow \end{pmatrix} + b \det \begin{pmatrix} \leftarrow \vec{A}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{B}_i \rightarrow \\ \vdots \\ \leftarrow \vec{A}_n \rightarrow \end{pmatrix}$$

Property ② implies that if two rows are equal, the determinant is zero.

We used the properties to show that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

**PROBLEM:** I am thinking of numbers $a, b, c, d, e, f$ such that

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ 1 & 1 & 1 \end{pmatrix} = -8 \quad \& \quad \det\begin{pmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{pmatrix} = 13.$$

Find $\det\begin{pmatrix} a & b & c \\ d & e & f \\ 4 & 7 & 10 \end{pmatrix}$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ 4 & 7 & 10 \end{pmatrix} = 1(-8) + 3(13) = 31.$$

$$\Uparrow$$

$$\begin{pmatrix} 4 & 7 & 10 \end{pmatrix} = 1\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

**PROBLEM:** If $A$ is an $n \times n$ matrix with $\det(A) = d$, what is $\det(cA)$?

$$c\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_n \end{pmatrix} = \begin{pmatrix} c\vec{A}_1 \\ \vdots \\ c\vec{A}_n \end{pmatrix}$$

$$\det(cA) = \det\begin{pmatrix} c\vec{A}_1 \\ \vdots \\ c\vec{A}_n \end{pmatrix} = c^n \det\begin{pmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_n \end{pmatrix} = c^n d.$$

pull $c$ out of each row, one at a time, by 3.

## §1. $3 \times 3$ matrices

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Let's begin with upper-triangular matrices:

$$|A| = \begin{vmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{vmatrix} = \begin{vmatrix} \alpha\vec{E}_1 + a\vec{E}_2 + b\vec{E}_3 \\ \beta\vec{E}_2 + c\vec{E}_3 \\ \gamma\vec{E}_3 \end{vmatrix}$$

linearity in row 1

$$= \alpha\begin{vmatrix} \vec{E}_1 \\ \beta\vec{E}_2 + c\vec{E}_3 \\ \gamma\vec{E}_3 \end{vmatrix} + a\begin{vmatrix} \vec{E}_2 \\ \beta\vec{E}_2 + c\vec{E}_3 \\ \gamma\vec{E}_3 \end{vmatrix} + b\begin{vmatrix} \vec{E}_3 \\ \beta\vec{E}_2 + c\vec{E}_3 \\ \gamma\vec{E}_3 \end{vmatrix}$$

linearity in rows 2 & 3

$$= \alpha\beta\gamma\begin{vmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{vmatrix} + \alpha c\gamma\begin{vmatrix} \vec{E}_1 \\ \vec{E}_3 \\ \vec{E}_3 \end{vmatrix} + a\beta\gamma\begin{vmatrix} \vec{E}_2 \\ \vec{E}_2 \\ \vec{E}_3 \end{vmatrix} + ac\gamma\begin{vmatrix} \vec{E}_2 \\ \vec{E}_3 \\ \vec{E}_3 \end{vmatrix} + b\beta\gamma\begin{vmatrix} \vec{E}_3 \\ \vec{E}_2 \\ \vec{E}_3 \end{vmatrix} + bc\gamma\begin{vmatrix} \vec{E}_3 \\ \vec{E}_3 \\ \vec{E}_3 \end{vmatrix}$$

all 0 due to repeated row

$$= \alpha\beta\gamma\,|I_3| = \alpha\beta\gamma.$$

The same calculation generalizes to give:

**Theorem:** If $A$ is an $n\times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

is the product of the diagonal entries.

Let $A = \begin{pmatrix}\alpha&\beta&\gamma\\a&b&c\\d&e&f\end{pmatrix}$. By linearity in row 1,

$$|A| \overset{(*)}{=} \alpha \begin{vmatrix}1&0&0\\a&b&c\\d&e&f\end{vmatrix} + \beta \begin{vmatrix}0&1&0\\a&b&c\\d&e&f\end{vmatrix} + \gamma\begin{vmatrix}0&0&1\\a&b&c\\d&e&f\end{vmatrix}.$$

Now notice that

$$\begin{vmatrix}1&0&0\\a&b&c\\d&e&f\end{vmatrix} = \begin{vmatrix}1&0&0\\a&b&c\\d&e&f\end{vmatrix} - a\overbrace{\begin{vmatrix}1&0&0\\1&0&0\\d&e&f\end{vmatrix}}^{=0} = \begin{vmatrix}1&0&0\\a&b&c\\d&e&f\end{vmatrix} - \begin{vmatrix}1&0&0\\a&0&0\\d&e&f\end{vmatrix}$$

$$= \begin{vmatrix}1&0&0\\0&b&c\\d&e&f\end{vmatrix} = \begin{vmatrix}1&0&0\\0&b&c\\d&e&f\end{vmatrix} - d\underbrace{\begin{vmatrix}1&0&0\\0&b&c\\1&0&0\end{vmatrix}}_{=0} = \begin{vmatrix}1&0&0\\0&b&c\\0&e&f\end{vmatrix}.$$

In this way, $(*)$ becomes

$$|A| = \alpha\begin{vmatrix}1&0&0\\0&b&c\\0&e&f\end{vmatrix} + \beta\begin{vmatrix}0&1&0\\a&0&c\\d&0&f\end{vmatrix} + \gamma\begin{vmatrix}0&0&1\\a&b&0\\d&e&0\end{vmatrix}.$$

Applying the same steps as for $n=2$ to the bottom 2 rows gives

1 swap — to get $I_3$ — 2 swaps

$$|A| = \alpha(bf-ec)\underbrace{\begin{vmatrix}1&&\\&1&\\&&1\end{vmatrix}}_{1} + \beta(af-cd)\underbrace{\begin{vmatrix}&1&\\1&&\\&&1\end{vmatrix}}_{-1} + \gamma(ae-bd)\underbrace{\begin{vmatrix}&&1\\1&&\\&1&\end{vmatrix}}_{1}$$

$$= \alpha\underbrace{\begin{vmatrix}b&c\\e&f\end{vmatrix}}_{A_{\hat{1}\hat{1}}} - \beta\underbrace{\begin{vmatrix}a&c\\d&f\end{vmatrix}}_{A_{\hat{1}\hat{2}}} + \gamma\underbrace{\begin{vmatrix}a&b\\d&e\end{vmatrix}}_{A_{\hat{1}\hat{3}}}$$

Here $A_{\hat{i}\hat{j}}$ means the $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting the $i^{th}$ row & $j^{th}$ column, and the formula we just derived is the Laplace expansion of $\det(A)$ along the first row.

Another sort of formula for $3\times 3$ is

$$\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}\begin{matrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\end{matrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

It is obtained by writing

$$\begin{vmatrix}\sum_{i=1}^3 a_{1i}\vec{E}_i\\ \sum_{j=1}^3 a_{2j}\vec{E}_j\\ \sum_{k=1}^3 a_{3k}\vec{E}_k\end{vmatrix} = \sum_{i,j,k=1}^3 a_{1i}a_{2j}a_{3k}\underbrace{\begin{vmatrix}\vec{E}_i\\\vec{E}_j\\\vec{E}_k\end{vmatrix}}_{=0,\,1,\text{ or }-1}$$

by multilinearity.

# §2. Laplace expansions

We first do the "expansion in a row":

**Theorem:** For any fixed $i$,

$$\det(A) = \sum_{j=1}^{n} a_{ij}\cdot \underbrace{(-1)^{i+j}\det(A_{\hat{ij}})}_{=:\, C_{ij} \,=\, (i,j)^{th}\ \text{cofactor}}$$

*Proof:* Let $A_{\hat{k}}(\vec{\beta})$ be the matrix obtained by replacing the $k^{th}$ row of $A$ by the row vector $\vec{\beta}$. Then

$$|A_{\hat{k}}(\vec{E}_\ell)| = \begin{array}{c} \\ \\ k^{th}\text{ row} \\ \\ \\ \end{array}\begin{vmatrix} a_{11} & \cdots & a_{1\ell} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\ell} & \cdots & a_{nn} \end{vmatrix} \underset{\substack{\text{swap } k^{th} \text{ row to top} \\ \text{past } (k-1) \text{ other rows}}}{=} (-1)^{k-1}\begin{vmatrix} 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & a_{1\ell} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\ell} & \cdots & a_{nn} \end{vmatrix}$$

$\ell^{th}$ column

omit the $a_{k*}$'s

argue as in $3\times3$ cases: "replace" operations don't change det

$$= (-1)^{k-1}\begin{vmatrix} 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} \underset{\substack{\text{swap } \ell^{th} \text{ column all the} \\ \text{way to left}}}{=} (-1)^{\ell-1+k-1}\left|\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_{\hat{k\ell}} & \\ 0 & & & \end{array}\right|$$

$$= (-1)^{k+\ell}\,|A_{\hat{k\ell}}|$$

where I am anticipating the result that determinants are also alternating and multi-linear in the columns. (The last step follows from the uniqueness of determinants since the $n\times n$ determinant of $\left|\begin{array}{c|c} 1 & \\ \hline & B \end{array}\right|$ satisfies the rules for $|B|$.)

Since the $k^{th}$ row of $A$ is

$$\vec{A}_k = \sum_{\ell=1}^{n} a_{k\ell}\vec{E}_\ell\,, \quad \text{linearity in this row} \Rightarrow$$

$$|A| = \sum_{\ell=1}^{n} a_{k\ell}\,|A_{\hat{k}}(\vec{E}_\ell)| = \sum_{\lambda=1}^{n}(-1)^{k+\ell} a_{k\ell}\,|A_{\hat{k\ell}}|.$$

$\square$

## §2. Laplace expansions

We first do the "expansion in a row":

**Theorem:** For any fixed $i$,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot \underbrace{(-1)^{i+j}\det(A_{\hat{i}\hat{j}})}_{=: C_{ij} = (i,j)^{th} \text{ cofactor}}$$

There is also the "expansion in a column":

for any fixed $j$,

$$\det(A) = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j} \det(A_{\hat{i}\hat{j}})$$

The proof is "the same", with the roles of rows & columns reversed.

Ex/
$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 3\left(\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}\right) - 1\left(\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

$$= 3(-3+1) - (1+1) = -8. \quad //$$

where I am anticipating the result that determinants are also alternating and multi-linear in the columns. (The last step follows from the uniqueness of determinants since the $n \times n$ determinant of $\begin{vmatrix} 1 & \\ & B \end{vmatrix}$ satisfies the rules for $|B|$.)

Since the $k^{th}$ row of $A$ is

$$\vec{A}_k = \sum_{l=1}^{n} a_{kl}\vec{E}_l, \quad \text{linearity in this row} \Rightarrow$$

$$|A| = \sum_{l=1}^{n} a_{kl}\left|A_{\hat{k}}(\vec{E}_l)\right| = \sum_{l=1}^{n} (-1)^{k+l} a_{kl}\left|A_{\hat{k}\hat{l}}\right|. \quad \square$$

Ex/
$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 3(-3) - 1 \cdot 1 + 1(9+1) + 2(-1-3)$$

$$= -8. \quad //$$

# §3. Determinants and EROs

Elementary Row Operations

Let $E$ be an elementary matrix, so that $\tilde{A} = EA$ is "one row operation applied to $A$".

**Theorem 1**: If $E$ is a $\begin{cases} \text{replace} \\ \text{swap} \\ \text{scale by } \mu \end{cases}$ operation, then $|\tilde{A}| = \begin{cases} |A| \\ -|A| \\ \mu|A|. \end{cases}$

**Proof**:

REPLACE
$$\underbrace{\begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j + \alpha\vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix}}_{|\tilde{A}|} = \underbrace{\begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_n \end{vmatrix}}_{|A|} + \alpha \underbrace{\begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix}}_{= 0 \text{ (repeated row)}}$$

SWAP
$$\begin{vmatrix} \vdots \\ \vec{A}_j \\ \vdots \\ \vec{A}_i \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_j \\ \vdots \end{vmatrix}$$

SCALE
$$\begin{vmatrix} \vec{A}_1 \\ \vdots \\ \mu\vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix} = \mu \begin{vmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_i \\ \vdots \\ \vec{A}_n \end{vmatrix}$$

$\square$

**PROBLEM:** Find $\begin{vmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{vmatrix}$ by row-reducing to an upper triangular matrix.

Without using scale or swap operations, we row-reduce to $\begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{vmatrix} = 9.$

**Theorem 2:** $\det A \neq 0 \iff A$ invertible

**Proof:** $(\Leftarrow)$: $A$ invertible $\Rightarrow$ $A$ is obtained from $I_n$ by EROs $\overset{\text{Thm. 1}}{\Longrightarrow}$

$\det A = \det I_n \cdot (-1)^{\# \text{swaps}} \times \begin{pmatrix} \text{product of} \\ \text{scaling factors} \end{pmatrix} \neq 0.$

$(\Rightarrow)$: $\det A \neq 0 \overset{\text{Thm. 1}}{\Longrightarrow}$

$\det(\operatorname{rref} A) = \det A \cdot (-1)^{\# \text{swaps}} \times \begin{pmatrix} \text{product of} \\ \text{scaling factors} \end{pmatrix} \neq 0$

$\underset{\text{HW}}{\Rightarrow}$ rref $A$ has no row of all zeroes

$\underset{A \text{ square}}{\Rightarrow}$ $\operatorname{rref} A = I_n \Rightarrow A$ invertible. $\square$

Ex/ We know $\operatorname{rref}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \neq I_3$, so

$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0.$ //

$\Rightarrow$ If $E$ is an elementary matrix, then $\det(EA) = \det E \cdot \det A$. $\quad (*)$

**Theorem 3:** Given $n \times n$ matrices $A$ & $B$,

$$\det AB = \det A \cdot \det B.$$

**Proof:** $\boxed{\text{Case 1}}$ $\det A = 0$. Then $A$ isn't invertible, so has rank $< n$ $\Rightarrow$ $AB$ has rank $< n$ $\Rightarrow$ $AB$ not invertible $\Rightarrow$ $\det AB = 0$.

$\boxed{\text{Case 2}}$ $\det A \neq 0$. Then $A$ is invertible, and so may be written as a product of elementary matrices: $A = E_N \cdots E_1 (I_n)$

$\Rightarrow AB = E_N \cdots E_1 B$. By repeated application of $(*)$,

$$\det A = \det E_N \cdots \det E_1 \cdot \cancel{\det I_n}\ 1$$

$$\det AB = \det E_N \cdots \det E_1 \cdot \det B. \qquad \square$$

**Corollary:** If $A$ is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Ex/ We know $\operatorname{rref}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \neq I_3$, so

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0. \quad //$$

Consider the elementary matrices once more. What are their determinants?

$\boxed{\text{REPLACE}}$ $$\begin{vmatrix} 1 & & \\ & \ddots & \\ \alpha & & 1 \end{vmatrix} = 1$$

$\boxed{\text{SWAP}}$ $$\begin{vmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & \cdots & 1 \\ & & 1 & \cdots & 0 \\ & & & & \ddots \\ & & & & & 1 \end{vmatrix} = -1$$

$\boxed{\text{SCALE}}$ $$\begin{vmatrix} 1 & & & \\ & \ddots & & \\ & & \mu & \\ & & & \ddots \\ & & & & 1 \end{vmatrix} = \mu$$

both upper/lower triangular (which includes diagonal)

Compare with:

**Theorem 1:** If $E$ is a $\begin{cases} \text{replace} \\ \text{swap} \\ \text{scale by } \mu \end{cases}$ operation, then $|EA| = \begin{cases} |A| \\ -|A| \\ \mu|A|. \end{cases}$

**Corollary**: Elementary column operations have the same effect on determinants as EROs.

Since the rows & columns of a matrix $A$ are independent if and only if $A$ is invertible, we also get a test for independence out of Theorem 2.

Ex/ For what values of $\alpha$ is $\begin{pmatrix}\alpha\\\alpha\\4\end{pmatrix}, \begin{pmatrix}1\\\alpha\\\alpha\end{pmatrix}, \begin{pmatrix}1\\1\\\alpha\end{pmatrix}$ a linearly independent set?

$$\begin{vmatrix}\alpha & 1 & 1\\ \alpha & \alpha & 1\\ 4 & \alpha & \alpha\end{vmatrix} = \begin{vmatrix}\alpha & 1 & 1\\ 0 & \alpha-1 & 0\\ 4 & \alpha & \alpha\end{vmatrix} = (\alpha-1)\begin{vmatrix}\alpha & 1\\ 4 & \alpha\end{vmatrix} = (\alpha-1)(\alpha^2-4)$$
$$= (\alpha-1)(\alpha-2)(\alpha+2)$$

$\Rightarrow$ this is <u>nonzero</u> if $\alpha \neq 1, 2, -2$.

Ex/ If $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 2$, find $\begin{vmatrix}a-8g & 3b-24h & c-8i\\ d & 3e & f\\ g & 3h & i\end{vmatrix}$.

$$= 3\begin{vmatrix}a-8g & b-8h & c-8i\\ d & e & f\\ g & h & i\end{vmatrix} = 3\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 3\cdot 2$$

linearity in 2nd column

row replace op.

$= 6.$ //

I have mentioned in a couple of places that if we used the columns instead of rows to define "det", we'd get the same function. (This was used in the proof of Laplace expansion.) Since transposing matrices takes rows to columns & vice versa, this statement is equivalent to

**Theorem 4**: $\det(A) = \det({}^tA)$.

**Proof**: $A$ & ${}^tA$ have the same rank, so are both invertible (with nonzero "det") or not. If they're both invertible, we have $A = E_N \cdots E_1$ & ${}^tA = {}^tE_1 \cdots {}^tE_N$.

By Theorem 3, it suffices to check that $\det(E_i) = \det({}^tE_i)$. But this is obvious: "swap" and "scale" matrices are unchanged by transpose, and "replace" matrices have determinant 1. $\square$

## §3. Determinants and Volume

Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$. What is the $n$-volume of the parallelepiped

$$P = P(\vec{v}_1, \ldots, \vec{v}_n) := \left\{ a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \;\middle|\; \begin{array}{c} 0 \le a_i \le 1 \\ \text{for each } i \end{array} \right\} ?$$

A basic observation is that this is unchanged by swapping $\vec{v}_i$ & $\vec{v}_j$, while if you scale (multiply) $\vec{v}_i$ by $\mu$, this multiplies the volume by $|\mu|$.

If you "replace" $\vec{v}_j$ by $\vec{v}_j + a\vec{v}_i$, this causes a shear of $P$, and shears don't affect volume.

These are precisely the effects that these operations have on $|\det(A)|$ (the absolute value of $\det(A)$), where

$$A = \begin{pmatrix} \leftarrow & \vec{v}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{v}_n & \rightarrow \end{pmatrix}.$$

More precisely, if a sequence of row operations gets you from $\vec{E}_1, \ldots, \vec{E}_n$ to $\vec{v}_1, \ldots, \vec{v}_n$ (as rows of a matrix), and involves scaling by $\mu_1, \ldots, \mu_\ell$ & $N$ swaps, then $\det A = (-1)^N \prod_{i=1}^{\ell} \mu_i$. Now

$\operatorname{vol}(P(\vec{E}_1, \ldots, \vec{E}_n)) = 1$, so

$\operatorname{vol}(P(\vec{v}_1, \ldots, \vec{v}_n)) = 1 \cdot \prod_{i=1}^{\ell} |\mu_i|$ by the above observations, and this is $|\det(A)|$.

**Theorem**: $\operatorname{Vol}\{P(\vec{v}_1, \ldots, \vec{v}_n)\} =$

$$\left| \det \begin{pmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{v}_n \rightarrow \end{pmatrix} \right| = \left| \det \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} \right|.$$

Now let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix $M$

**Corollary**: If $\Omega \subset \mathbb{R}^n$ is any region,

$$\frac{\operatorname{Vol}\{T(\Omega)\}}{\operatorname{Vol}\{\Omega\}} = |\det(M)|$$

**Sketch of Proof**: Covering $\Omega$ with little parallelepipeds and taking a limit as their size $\to 0$, we see that it suffices to check the result for such parallelepipeds.

$$\frac{\operatorname{Vol}\{T(P(\vec{v}_1, \ldots, \vec{v}_n))\}}{\operatorname{Vol}\{P(\vec{v}_1, \ldots, \vec{v}_n)\}} = \frac{\operatorname{Vol}\{P(T\vec{v}_1, \ldots, T\vec{v}_n)\}}{\operatorname{Vol}\{P(\vec{v}_1, \ldots, \vec{v}_n)\}}$$

$$= \left| \frac{\det \begin{pmatrix} \uparrow & & \uparrow \\ M\vec{v}_1 & \cdots & M\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}}{\det \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}} \right| = \left| \frac{\det \left( M \cdot \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} \right)}{\det \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}} \right|$$

$$= \left| \frac{\det M \cdot \cancel{\det \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}}}{\cancel{\det \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}}} \right| = |\det M|. \quad \square$$

Ex/ $\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \le 1 \right\}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ has matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$\Rightarrow T(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$

$\operatorname{area}(T(\Omega)) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \operatorname{area}(\Omega) = ab\pi$. //