Determinants
Today we'll continue our brief study of determinants, the goal of which is to prepare for the change of variable formula in multiple integrals. At the end of the last lecture, we proved that there is (for each n) exactly one function

\[ \text{det} : \{ n \times n \text{ matrices} \} \rightarrow \mathbb{R} \]

satisfying these properties:

1. **Normalization:**
   \[ \text{det} \left( I_n \right) = 1 \]
   (n \times n \text{ identity matrix})

2. **Antisymmetry in the rows:**
   If two rows are swapped and all the other rows stay the same, the determinant gets multiplied by -1.

3. **Multilinearity in the rows:**
   \[
   \text{det} \left( \begin{array}{c}
   \vec{A}_1 \\
   \vdots \\
   \vec{A}_n
   \end{array} \right) = \text{a \ det} \left( \begin{array}{c}
   \vec{A}_1 \\
   \vdots \\
   \vec{B}_i \\
   \vec{A}_n
   \end{array} \right) + \text{b \ det} \left( \begin{array}{c}
   \vec{A}_1 \\
   \vdots \\
   \vec{B}_i \\
   \vec{A}_n
   \end{array} \right)
   \]

Property 2 implies that if two rows are equal, the determinant is zero. We used the properties to show that \( \text{det} \left( \begin{array}{cc}
   a & b \\
   c & d
   \end{array} \right) = ad - bc \).
**Problem:** I am thinking of numbers $a, b, c, d, e, f$ such that

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  1 & 1 & 1 \\
\end{vmatrix} = -8 \quad \text{and} \quad \begin{vmatrix}
  a & b & c \\
  d & e & f \\
  1 & 2 & 3 \\
\end{vmatrix} = 13.
\]

Find \[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  4 & 7 & 10 \\
\end{vmatrix}
\]

\[
= 1(-8) + 3(13)
\]

\[
= 31.
\]

\[
(4 \ 7 \ 10) = 1(1 \ 1 \ 1) + 3(1 \ 2 \ 3)
\]

**Problem:** If $A$ is an $n \times n$ matrix with $\det(A) = d$, what is $\det(cA)$? Let

\[
\begin{pmatrix}
  \hat{A}_1 \\
  \vdots \\
  \hat{A}_n \\
\end{pmatrix} = \begin{pmatrix}
  \hat{A}_1 \\
  \vdots \\
  \hat{A}_n \\
\end{pmatrix}
\]

\[
\begin{vmatrix}
  c\hat{A}_1 \\
  \vdots \\
  c\hat{A}_n \\
\end{vmatrix} = c^n \begin{vmatrix}
  \hat{A}_1 \\
  \vdots \\
  \hat{A}_n \\
\end{vmatrix} = c^n d.
\]

Pull $c$ out of each row one at a time, by (3).
3.1. $3 \times 3$ matrices

Let's begin with upper-triangular matrices:

$$|A| = \begin{vmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{vmatrix} = abc = \sum_{i=1}^{3} (-1)^{i+1} a_i E_i$$

linearity in row 1

$$= a \begin{vmatrix} E_1 \end{vmatrix} + b \begin{vmatrix} E_2 \end{vmatrix} + c \begin{vmatrix} E_3 \end{vmatrix}$$

linearity in rows 2 & 3

$$= a \beta^1 + b \beta^2 + c \beta^3$$

linearity in rows 1 & 2 & 3

$$= \alpha \beta^1 + \beta \beta^2 + \gamma \beta^3$$

The same calculation generalizes to give:

**Theorem:** If $A$ is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11} a_{22} \cdots a_{nn}$$

is the product of the diagonal entries.
Let \( A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \). By linearity in row 1,

\[
|A| = x \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ d & e & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ d & e & f \end{vmatrix}.
\]

Now notice that

\[
\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix} \begin{vmatrix} 0 \\ c \end{vmatrix} = 0,
\]

\[
\begin{vmatrix} 0 & 1 & 0 \\ d & e & f \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ d & e \end{vmatrix} \begin{vmatrix} 0 \\ f \end{vmatrix} = 0,
\]

\[
\begin{vmatrix} 0 & 0 & 1 \\ d & e & f \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ d & e \end{vmatrix} \begin{vmatrix} 1 \\ f \end{vmatrix} = 0.
\]

In this way, (4) becomes

\[
|A| = a \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ 0 & e & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ 0 & e & f \end{vmatrix}.
\]

Applying the same steps as for \( n = 2 \) to the bottom 2 rows gives

\[
|A| = 0 \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} - \beta \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \gamma \begin{vmatrix} a & b \\ d & e \end{vmatrix}.
\]

Here \( A_{ij} \) means the \((n-1) \times (n-1)\) matrix obtained from \( A \) by deleting the \( i \)th row and \( j \)th column, and the formula we just derived is the Laplace expansion of \( \det(A) \) along the first row.

Another sort of formula for \( 3 \times 3 \) is

\[
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{32} - a_{11}a_{23}a_{32}.
\]

It is obtained by writing

\[
\begin{vmatrix} \Sigma_{i=1}^3 a_{1i} E_i \\ \Sigma_{j=1}^3 a_{2j} E_j \\ \Sigma_{k=1}^3 a_{3k} E_k \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} E_1 \\ E_2 \\ E_3 \end{vmatrix} = \begin{vmatrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{23}a_{32} - a_{11}a_{23}a_{32} \end{vmatrix} E_1.
\]

by multilinearity.

\[
= 0, 1, \text{or} -1
\]
2.2. Laplace expansions

We first do the "expansion in a row":

**Theorem:** For any fixed $i$,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \det(A_{ij})$$

where $C_{ij} = (i,j)^{th}$ cofactor

**Proof:** Let $A_k^k(\vec{\beta})$ be the matrix obtained by replacing the $k^{th}$ row of $A$ by the row vector $\vec{\beta}$. Then

$$|A_k^k(\vec{E}_k)| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kn} \end{vmatrix} = (-1)^{k-1} a_{11} \ldots a_{k-1} 0 \ldots a_{nn}$$

Sweep the $k^{th}$ row to top post $(k-1)$ other rows.

Sweep the $k^{th}$ column on all the $a_{kx}$'s.

$$\begin{vmatrix} 0 & 1 & \ldots & 0 \\ a_{11} & 0 & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix} = (-1)^{k-1} a_{11} \ldots a_{k-1} 0 \ldots a_{nn}$$

$$\begin{vmatrix} 0 & 1 & \ldots & 0 \\ a_{11} & 0 & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kn} \end{vmatrix}$$

$$\begin{vmatrix} 0 & 1 & \ldots & 0 \\ a_{11} & 0 & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix}$$

argue as in $3 \times 3$ cases:

"replace operations don't change det"

$$= (-1)^{k-1} |A_k^k(\vec{E}_k)|$$

The last step follows from the uniqueness of determinants since the non-determinant of $|B|$ satisfies the rules for $|B|$.

Since the $k^{th}$ row of $A$ is

$$A_k^k = \sum_{k=1}^{n} a_{kx} \vec{E}_x$$

linearity in this row $\Rightarrow$

$$|A| = \sum_{k=1}^{n} a_{kx} |A_k^k(\vec{E}_k)| = \sum_{k=1}^{n} (-1)^{k+1} a_{kx} |A_k^k|.$$  $\square$
22. Laplace expansions

We first do the "expansion in a row":

**Theorem:** For any fixed $i$,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \det(A_{ij})$$

where $i$ is the $(i)^{th}$ column.

There is also the "expansion in a column":

for any fixed $j$,

$$\det(A) = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j} \det(A_{ij})$$

The proof is "the same", with the roles of rows & columns reversed.

Example:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 3(-3) - 1(1+1) = -8.$$
33. Determinants and EROs

Let $E$ be an elementary matrix, so that $\tilde{A} = EA$ is "one row operation applied to $A$".

**Theorem:** If $E$ is a {replace, swap operation, scale by $\mu$}, then $|\tilde{A}| = \begin{cases} |A| \\
-|A| \\
\mu|A| \end{cases}$.

**Proof:**

**REPLACE**

\[
\begin{vmatrix} A_1 \\
\vdots \\
A_i + \mu A_j \\
\vdots \\
A_n \end{vmatrix} = \begin{vmatrix} A_1 \\
\vdots \\
A_i \\
\vdots \\
A_n \end{vmatrix} + \mu \begin{vmatrix} A_1 \\
\vdots \\
A_j \\
\vdots \\
A_n \end{vmatrix}
\]

**SWAP**

\[
\begin{vmatrix} A_j \\
\vdots \\
A_i \\
\vdots \\
A_n \end{vmatrix} = -\begin{vmatrix} A_i \\
\vdots \\
A_j \\
\vdots \\
A_n \end{vmatrix}
\]

**SCALE**

\[
\begin{vmatrix} \lambda A_i \\
\vdots \\
A_n \end{vmatrix} = \lambda \begin{vmatrix} A_i \\
\vdots \\
A_n \end{vmatrix}
\]

$= 0$ (repeated row)
Problem: Find \[
\begin{vmatrix}
1 & -1 & 2 & -2 \\
-1 & 2 & 1 & 4 \\
2 & 1 & 14 & 10 \\
-2 & 6 & 10 & 33
\end{vmatrix}
\]
by row-reducing to an upper triangular matrix.

Without using scale or swap operations, we row-reduce to \[
\begin{vmatrix}
1 & -1 & 2 & -2 \\
0 & 1 & 3 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 9
\end{vmatrix}
= 9.
\]

**Theorem 2:** \(\det A \neq 0 \iff A\) is invertible.

**Proof:**

(\(\implies\)): A invertible \(\implies A\) is obtained from \(I_n\) by EROs \(\Rightarrow\)
\[
\det A = \det I_n \cdot (-1)^{\text{swaps}} \times (\text{product of scaling factors}) \neq 0.
\]

(\(\impliedby\)): \(\det A \neq 0 \implies \)
\[
\det (\text{rref } A) = \det A \cdot (-1)^{\text{swaps}} \times (\text{product of scaling factors}) \neq 0
\]

\(\implies\) rref \(A\) has no row of all zeros

\(\implies A = \text{rref } A = I_n \implies A\) is invertible. \(\square\)

Ex/ We know \(\text{rref } (\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}) \neq I_3\), so \[
\det (\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}) = 0.
\]
\[ \text{Theorem 3: Given } n \times n \text{ matrices } A \text{ and } B \]
\[ \det(AB) = \det A \cdot \det B. \]

**Proof:**

**Case 1** \( \det A = 0 \). Then \( A \) isn't invertible, so has rank \( < n \) \( \Rightarrow \) \( AB \) has rank \( < n \) \( \Rightarrow \) \( AB \) not invertible \( \Rightarrow \) \( \det AB = 0 \).

**Case 2** \( \det A \neq 0 \). Then \( A \) is invertible, and so may be written as a product of elementary matrices: \( A = E_n \cdots E_1(I_n) \)

\( \Rightarrow AB = E_n \cdots E_1B. \) By repeated application of (\#),

\( \det A = \det E_n \cdots \det E_1 \cdot \det A \)
\( \det AB = \det E_n \cdots \det E_1 \cdot \det B. \)

\( \square \)

**Corollary:** If \( A \) is invertible, then

\( \det (A^{-1}) = \frac{1}{\det(A)}. \)

**Example:** We know \( \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \neq I_3 \), so

\( \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0. \)

Consider the elementary matrices once more. What are their determinants?

- **Replace**

  \( \begin{vmatrix} 1 \end{vmatrix} = 1 \)

- **Swap**

  \( \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \)

- **Scale**

  \( \begin{vmatrix} \lambda \end{vmatrix} = \lambda \)

**Compare**

Theorem: If \( E \) is a \{replace, swap, scale\} operation, then \( \det EA = \begin{cases} |A| & \text{replace} \\ -|A| & \text{swap} \\ \mu |A| & \text{scale by } \mu \end{cases} \).
Corollary: Elementary column operations have the same effect on determinants as EROs.

Since the rows & columns of a matrix $A$ are independent if and only if $A$ is invertible, we also get a test for independence out of Theorem 2.

**Example:** For what values of $x$ is $(\frac{x}{4}), (\frac{x}{x}), (\frac{1}{x})$ a linearly independent set?

\[
\begin{vmatrix}
 x & 1 & 1 \\
 x & 0 & 1 \\
 x & 1 & 0
\end{vmatrix} = (x-1) x
\]

\[
\begin{vmatrix}
 0 & 1 & 1 \\
 x & 1 & 0 \\
 x & 1 & 0
\end{vmatrix} = (x-1)(x^2 - 1)
\]

⇒ this is nonzero if $x \neq 1, 2, -2$.

**Example:** If \(
\begin{vmatrix}
 a & b & c \\
 d & e & f \\
 g & h & i
\end{vmatrix}
\)

is 2, find \(
\begin{vmatrix}
 a-8g & 3b-29h & c-8i \\
 d & 3e & f \\
 g & 3h & i
\end{vmatrix}
\)

= 8 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \cdot 2

Therefore, by linearity in 2nd column, row replace op.

I have mentioned in a couple of places that if we used the columns instead of rows to define "det", we'd get the same function. (This was used in the proof of Laplace expansion.) Since transposing matrices takes rows to columns & vice versa, this statement is equivalent to

**Theorem 4:** \(\det(A) = \det(^tA)\).

**Proof:** $A$ & $^tA$ have the same rank, so are both invertible (with nonzero "det") or not. If they're both invertible, we have $A = E_n \cdots E_1 \& ^tA = ^tE_1 \cdots ^tE_n$. By Theorem 3, it suffices to check that $\det(E_i) = \det(^tE_i)$. But this is obvious: "swap" and "scale" matrices are unchanged by transpose, and "replace" matrices have determinant 1.\\

\[\square\]
3. Determinants and Volume

Let $v_1, \ldots, v_n \in \mathbb{R}^n$. What is the $n$-volume of the parallelepiped $P = P(v_1, \ldots, v_n) := \{a_1v_1 + \cdots + a_nv_n \mid 0 \leq a_i \leq 1 \text{ for each } i \}$?

A basic observation is that this is unchanged by swapping $v_i$ and $v_j$, while if you scale (multiply) $v_i$ by $\mu$, this multiplies the volume by $|\mu|$.

If you "replace" $v_j$ by $v_j + a_i v_i$, this causes a shear of $P$, and shears don't affect volume.

These are precisely the effects that these operations have on $|\text{det}(A)|$ (the absolute value of $\text{det}(A)$), where

$A = \begin{pmatrix} \vec{v_1} \\ \vdots \\ \vec{v_n} \end{pmatrix}$. 
More precisely, if a sequence of row operations gets you from \( \tilde{E}_1, \ldots, \tilde{E}_n \) to \( \tilde{v}_1, \ldots, \tilde{v}_n \) (as rows of a matrix), and involves scaling by \( m_1, \ldots, m_k \) and \( N \) swaps, then 
\[
\text{det} A = (-1)^N \prod_{i=1}^N (m_i).
\]
Now 
\[
\text{vol}(P(\tilde{E}_1, \ldots, \tilde{E}_n)) = 1,
\]
so 
\[
\text{vol}(P(\tilde{v}_1, \ldots, \tilde{v}_n)) = 1 \cdot \prod_{i=1}^N (m_i) \text{vol} \text{ by the above observations, and thus is } |\text{det}(A)|.
\]

**Theorem:** 
\[
\text{Vol}\{P(\tilde{v}_1, \ldots, \tilde{v}_n)\} = \left| \begin{array}{c} \text{det} \left( \begin{array}{c} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{array} \right) \end{array} \right| = \left| \begin{array}{c} \text{det} \left( \begin{array}{c} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{array} \right) \end{array} \right|.
\]

Now let \( T: \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation with matrix \( M \).

**Corollary:** If \( \mathcal{S} \subset \mathbb{R}^n \) is any region,
\[
\frac{\text{Vol}\{T(\mathcal{S})\}}{\text{Vol}\{\mathcal{S}\}} = |\text{det}(M)|
\]

**Sketch of Proof:** Covering \( \mathcal{S} \) with little parallelepipeds and taking a limit as their size \( \to 0 \), we see that it suffices to check the result for such parallelepipeds.

\[
\frac{\text{Vol}\{T(\mathcal{S}))\}}{\text{Vol}\{\mathcal{S}\}} = \frac{\text{Vol}\{\mathcal{S})\}}{\text{Vol}\{\mathcal{S}\}} = \left| \begin{array}{c} \text{det} \left( \begin{array}{c} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{array} \right) \end{array} \right| = \left| \begin{array}{c} \text{det} \left( \begin{array}{c} \tilde{v}_1 \\ \vdots \\ \tilde{v}_n \end{array} \right) \end{array} \right| = |\text{det} M|.
\]

**Example:** \( \mathcal{S} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \)

\( T: \mathbb{R}^2 \to \mathbb{R}^2 \) has matrix \((a, b)\)
\( T(\mathcal{S}) = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\} \)
\( \text{area}(T(\mathcal{S})) = |a \cdot b| \text{area}(\mathcal{S}) = ab \pi. \)