CHANGE OF VARIABLES

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Though I usually don't write them out, there have been a lot of substitutions in the 1-variable integrals we compute. So let's recall how this goes:

given

$$u \longmapsto g(u) = x \qquad C^1 \text{ function}$$

mapping

$$[c,d] \longmapsto [a,b] \qquad a = g(c), \ b = g(d)$$

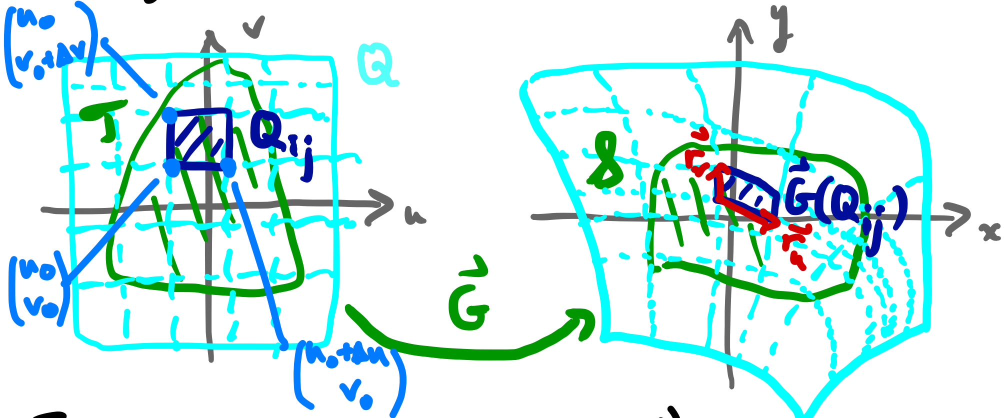
in 1-to-1 fashion, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\, g'(u)\, du$$

provided both integrals exist. We are after a multivariable generalization of this, starting with two variables.

§1. Change of variables $(n=2)$

We give an informal explanation of how (*) is obtained. Enclose T in a rectangle and partition it:



For simplicity we assume $\vec G$ extends to Q. The $\vec G(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross-product:

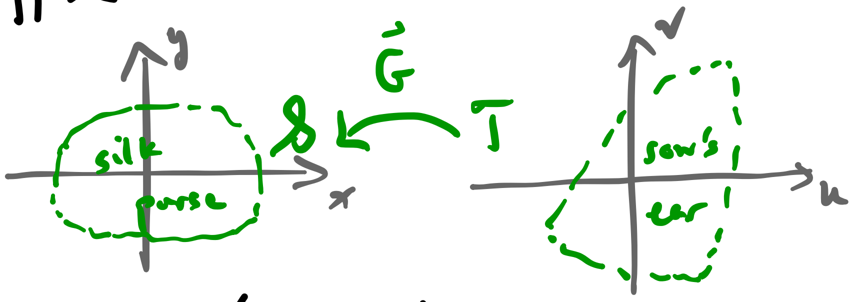
$$\vec r_u := \vec G\begin{pmatrix}u_0+\Delta u\\ v_0\end{pmatrix} - \vec G\begin{pmatrix}u_0\\ v_0\end{pmatrix}$$

$$\vec r_v := \vec G\begin{pmatrix}u_0\\ v_0+\Delta v\end{pmatrix} - \vec G\begin{pmatrix}u_0\\ v_0\end{pmatrix}$$

$$\text{Area}(\vec G(Q_{ij})) \approx \|\vec r_u \times \vec r_v\|$$

§1. Change of variables $(n=2)$

Suppose



is a 1-1/onto C^1 map. We'll write

$$\begin{pmatrix}u\\ v\end{pmatrix} \longmapsto \vec G\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}x\binom{u}{v}\\ y\binom{u}{v}\end{pmatrix}$$

and not worry about $\vec G$ failing to be 1-1 on the boundary or more generally a content-zero subset. Then for any continuous function $f: S \to \mathbb{R}$,

$$\boxed{\int_S \underbrace{f\,dA}_{dx\,dy} \overset{(*)}{=} \int_T (f\circ\vec G)\cdot \underset{\text{“}\frac{\partial(x,y)}{\partial(u,v)}\text{”}}{|\det D\vec G|}\, \underbrace{dA}_{du\,dv}.}$$

This is far from a rigorous proof!

I'll outline how Shifrin gets there later.

We'll do some examples now:

Ex/ $\vec{G}\begin{pmatrix} r \\ \theta \end{pmatrix} := \begin{pmatrix} x\begin{pmatrix} r \\ \theta \end{pmatrix} \\ y\begin{pmatrix} r \\ \theta \end{pmatrix} \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$$\Rightarrow D\vec{G} = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\Rightarrow \left|\det D\vec{G}\right| = r(\cos^2\theta + \sin^2\theta) = r$$

$$\Rightarrow \int_{\substack{S \\ \shortparallel \\ \vec{G}(T)}} f\begin{pmatrix} x \\ y \end{pmatrix} dx\,dy = \int_T (f\circ\vec{G})\begin{pmatrix} u \\ v \end{pmatrix} \left|\det D\vec{G}\right| dr\,d\theta$$

$$= \int_T f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} r\,dr\,d\theta.$$

//

Or more explicitly:

$$\vec{r}_u \approx \underbrace{\begin{pmatrix} \frac{\partial x}{\partial u}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ \frac{\partial y}{\partial u}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{pmatrix} \Delta u}, \quad \vec{r}_v \approx \underbrace{\begin{pmatrix} \frac{\partial x}{\partial v}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \\ \frac{\partial y}{\partial v}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{pmatrix} \Delta v}$$

$$\text{Area}(\vec{G}(Q_{ij})) \approx \left\| \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \boxed{\frac{\partial x}{\partial u}\Delta u \quad \frac{\partial y}{\partial u}\Delta u \quad 0} \\ \boxed{\frac{\partial x}{\partial v}\Delta v \quad \frac{\partial y}{\partial v}\Delta v \quad 0} \end{vmatrix} \right\|$$

$$= \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| \Delta u\,\Delta v$$

$$= \left|\det\left(D\vec{G}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right)\right| \cdot \text{Area}(Q_{ij})$$

which is consistent with the "determinant as expansion factor" result of Lecture 20. So

point in Q_{ij}

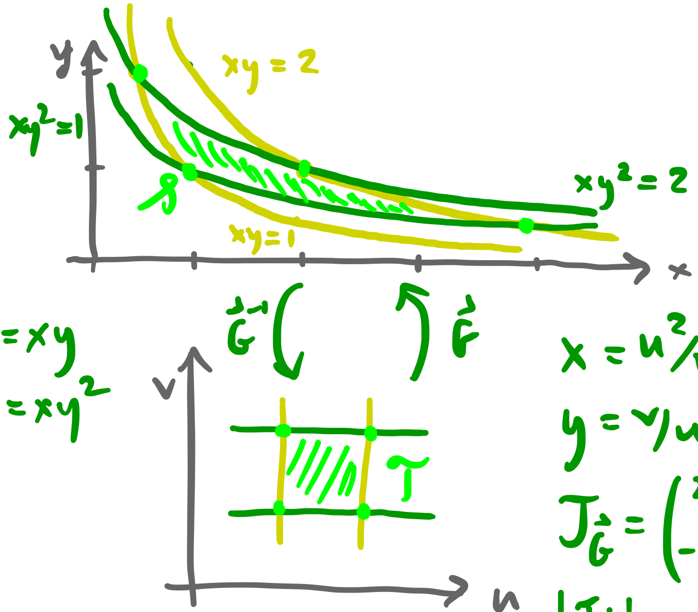
$$\int_S f\,dA = \lim_{\boxed{\|P\|\to 0}} \sum_{i,j} f\left(\vec{G}\left(\bigcirc\!\!\!\!\vec{q}_{ij}\right)\right) \underbrace{\text{Area}\left(\vec{G}(Q_{ij})\right)}_{\approx}$$

ie, as side-lengths of the Q_{ij} go to zero

$$= \lim_{\|P\|\to 0} \sum_{i,j} (f\circ\vec{G})(\vec{q}_{ij}) \overbrace{\left|\det D\vec{G}(\vec{q}_{ij})\right| \text{Area}(Q_{ij})}$$

$$= \int_T (f\circ\vec{G}) \left|\det D\vec{G}\right| dA.$$

PROBLEM: Calculate $\iint_S y^2\,dA$:



$u = xy$
$v = xy^2$

$x = u^2/v$

$y = v/u$

$J_{\vec{G}} = \begin{pmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{pmatrix}$

$|J_{\vec{G}}| = \frac{2}{v} - \frac{1}{v} = \frac{1}{v} > 0$

$$\iint_S y^2\,dx\,dy = \iint_T \left(\frac{v}{u}\right)^2 \boxed{\frac{1}{v}}\,du\,dv$$

$$= \int_1^2\int_1^2 \frac{v}{u^2}\,du\,dv$$

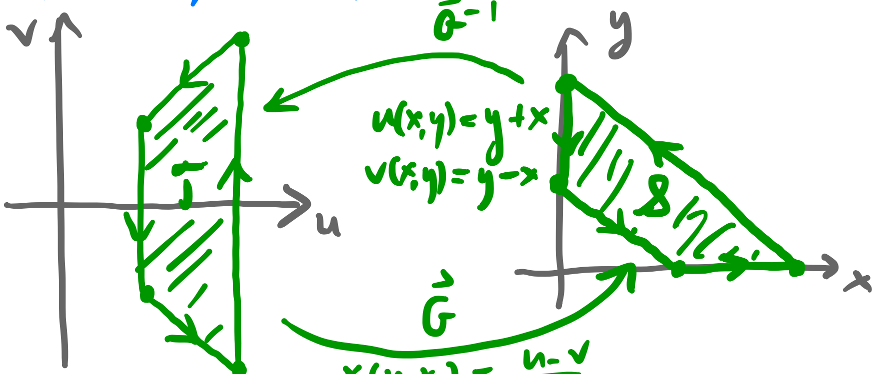
$$= \left(\int_1^2 v\,dv\right)\left(\int_1^2 \frac{1}{u^2}\,du\right)$$

$$= \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}.$$

//

Ex/ Compute $\iint_S \cos\left(\frac{\overset{v}{\boxed{y-x}}}{\underset{u}{\boxed{y+x}}}\right) dA$,

where S is the quadrilateral with vertices $(1,0)$, $(2,0)$, $(0,2)$, & $(0,1)$ $(=(x,y))$.



$u(x,y) = y + x$
$v(x,y) = y - x$

$x(u,v) = \frac{u-v}{2}$
$y(u,v) = \frac{u+v}{2}$

$$\Rightarrow J_{\vec{G}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \Rightarrow |J_{\vec{G}}| = \frac{1}{2}.$$

$$\Rightarrow \iint_S \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_T \cos\left(\frac{v}{u}\right)\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_1^2\int_{-u}^{u} \cos\left(\frac{v}{u}\right) dv\,du = \frac{1}{2}\int_1^2 \left[u\sin\left(\frac{v}{u}\right)\right]_{v=-u}^{v=u} du$$

$$= \frac{1}{2}\int_1^2 \{u\sin(1) - u\sin(-1)\}\,du = \sin(1)\int_1^2 u\,du$$

$$= \frac{3}{2}\sin(1).$$

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§2. Change of Variable (any n)

Suppose we want to integrate a function f on a region $\mathcal{S} \subset \mathbb{R}^n$, using a change of variable $\vec{G}$.

That is, there is an open set $\mathcal{U} \subset \mathbb{R}^n$, a C^1 function $\vec{G} : \mathcal{U} \to \mathbb{R}^n$ which (except possibly on a subset of content 0) is 1-1 with invertible derivative $D\vec{G}$, a region $\tilde{T} \subset \mathcal{U}$ with $\vec{G}(\tilde{T}) = \mathcal{S}$, and $\begin{cases} f \\ (f \circ \vec{G})|\det D\vec{G}| \end{cases}$ is integrable on $\begin{cases} \mathcal{S} \\ \tilde{T} \end{cases}$.

Then we have the equality

$$\int_{\underset{\vec{G}(\tilde{T})}{\overset{\shortparallel}{\mathcal{S}}}} f(\vec{x}) \underset{dx_1 \cdots dx_n}{dV} = \int_{\tilde{T}} f(\vec{G}(\vec{u})) \left| \det D\vec{G}(\vec{u}) \right| \underset{du_1 \cdots du_n}{dV}.$$

This is the "Change of Variable Theorem"

Ex 2/ $\vec{G}\begin{pmatrix} r \\ \theta \\ w \end{pmatrix} := \begin{pmatrix} x\begin{pmatrix} r \\ \theta \\ w \end{pmatrix} \\ y\begin{pmatrix} r \\ \theta \\ w \end{pmatrix} \\ z\begin{pmatrix} r \\ \theta \\ w \end{pmatrix} \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ w \end{pmatrix}$

$$D\vec{G} = \begin{pmatrix} x_r & x_\theta & x_w \\ y_r & y_\theta & y_w \\ z_r & z_\theta & z_w \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow |\det D\vec{G}| = r.$ This recovers

cylindrical integration

$$\int_{\mathcal{S}} f \, dx\,dy\,dz = \int_{\vec{G}^{-1}(\mathcal{S})} f\circ\vec{G} \; r\,dr\,d\theta\,dz. \quad //$$

PROBLEM: $\vec{G}\begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} := \begin{pmatrix} x\begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} \\ y\begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} \\ z\begin{pmatrix} \rho \\ \phi \\ \theta \end{pmatrix} \end{pmatrix} := \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix}$

$$\Rightarrow \quad D\vec{G} = \begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix} = \left(\begin{array}{c:c:c} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \hdashline \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \hdashline \cos\phi & -\rho\sin\phi & 0 \end{array}\right)$$

$$\det D\vec{G} = \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \rho^2\cos^2\phi\sin\phi \underbrace{\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}}_{1} + \rho^2\sin^2\phi\sin\phi \underbrace{\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}}_{1}$$

$$= \rho^2\sin\phi$$

Which recovers the spherical integration formula. //

Ex 1/ Say $\vec{G}: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with matrix $\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$,

(assume $c_i > 0$)

and $\mathcal{T}$ is a rectangle $[a_1,b_1]\times\cdots\times[a_n,b_n]$. Then $\mathcal{S} = [c_1a_1, c_1b_1]\times\cdots\times[c_na_n, c_nb_n]$ and we calculate

$$\int_{\mathcal{S}} f\,dV \overset{\text{Fubini}}{=} \int_{c_na_n}^{c_nb_n}\cdots\int_{c_1a_1}^{c_1b_1} f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} dx_1\cdots dx_n$$

$x_i := c_iu_i$

$dx_i = c_i\,du_i$

$$= c_1\cdots c_n \int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f\begin{pmatrix} c_1u_1 \\ \vdots \\ c_nu_n \end{pmatrix} du_1\cdots du_n$$

$$= \left|\det\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}\right| \int_{\mathcal{T}} f(\vec{G}(\vec{u}))\,dV,$$

which proves the Theorem in this case. //

$$\boxed{\int_{\vec{G}(\mathcal{T})} f(\vec{x}) \underset{dx_1\cdots dx_n}{dV} = \int_{\mathcal{T}} f(\vec{G}(\vec{u}))\,|\det D\vec{G}(\vec{u})| \underset{du_1\cdots du_n}{dV}.}$$

The key to the proof of the formula

$$\boxed{\int_{\vec{G}(\vec{J})} f(\vec{x})\, \underset{dx_1 \cdots dx_n}{dV} = \int_{\vec{J}} f(\vec{G}(\vec{u}))\, \left|\det D\vec{G}(\vec{u})\right| \underset{du_1 \cdots du_n}{dV}.}$$

is the following:

open subset in $\mathbb{R}^n$

Lemma: Suppose $\vec{G} : \mathcal{U} \to \mathbb{R}^n$ is C^1, $D\vec{G}$ is invertible, and $C_r \subset \mathcal{U}$ is a cube of width r centered at $\vec{a}$, on which

$$\| D\vec{G}(\vec{a})^{-1} \cdot D\vec{G}(\vec{x}) - I \|_{\square} < \epsilon < 1 \quad (\forall\, \vec{x} \in C_r).$$

Then $\vec{G}(C_r)$ is a region, and

$$(1-\epsilon)^n \tau V \le \mathrm{vol}(\vec{G}(C_r)) \le (1+\epsilon)^n \tau V$$

where $\tau := |\det D\vec{G}(\vec{a})|$ and $V := \mathrm{vol}(C_r)$.

Here $\|\vec{x}\|_{\square} := \max\{|x_1|, \ldots, |x_n|\}$.

So if $D\vec{G}$ doesn't change much on C_r, then

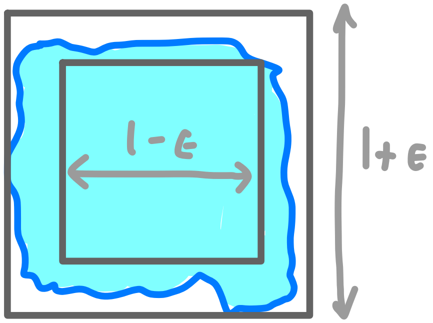
$$(1-\epsilon)^n \left|\det D\vec{G}(\vec{a})\right| \le \frac{\mathrm{vol}(\vec{G}(C_r))}{\mathrm{vol}(C_r)} \le (1+\epsilon)^n \left|\det D\vec{G}(\vec{a})\right|$$

is much as if we had applied a linear transformation.

Sketch: By a linear change of coordinates, we can reduce to the case $\vec{a}=\vec{0}$, $D\vec{G}(\vec{a})=I$ $(\Rightarrow \tau=1)$, and $r=1$ $(\Rightarrow V=1)$. So we must show $(1-\epsilon)^n \le \mathrm{vol}(\vec{G}(C_1)) \le (1+\epsilon)^n$.

This will follow if we can show that

$$C_{1-\epsilon} \subseteq \vec{G}(C_1) \subseteq C_{1+\epsilon}$$



which in turn means showing that

(1) $\|\vec{y}\|_\square \le \frac{1-\epsilon}{2} \Rightarrow \vec{y} \in \vec{G}(C_1)$, and

(2) $\vec{x} \in C_1 \Rightarrow \|\vec{G}(\vec{x})\|_\square \le \frac{1+\epsilon}{2}$.

Assertion (2) follows from the mean value inequality:

$$\|\vec{G}(\vec{x})\|_\square \le \Big(\max_{\vec{c}\in[\vec{0},\vec{x}]} \underbrace{\|D\vec{G}(\vec{c})\|_\square}_{\le \|D\vec{G}-I\|_\square + \|I\|_\square < \epsilon+1}\Big)\overbrace{\|\vec{x}\|_\square}^{\le 1/2}$$

Combining this argument with the proof of the inverse function thm. gives (1).

The key to the proof of the formula

$$\boxed{\int_{\vec{G}(\vec{J})} f(\vec{x})\, \underset{dx_1\cdots dx_n}{dV} = \int_{\vec{J}} f(\vec{G}(\vec{u}))\,|\det D\vec{G}(\vec{u})|\, \underset{du_1\cdots du_n}{dV}.}$$

is the following:

open subset in $\mathbb{R}^n$

Lemma: Suppose $\vec{G}: \mathcal{U} \to \mathbb{R}^n$ is C^1, $D\vec{G}$ is invertible, and $C_r \subset \mathcal{U}$ is a cube of width r centered at $\vec{a}$, on which $\|D\vec{G}(\vec{a})^{-1} \cdot D\vec{G}(\vec{x}) - I\|_\square < \epsilon < 1$ $(\forall \vec{x} \in C_r)$.
Then $\vec{G}(C_r)$ is a region, and

$$(1-\epsilon)^n \tau V \le \mathrm{vol}(\vec{G}(C_r)) \le (1+\epsilon)^n \tau V$$

where $\tau := |\det D\vec{G}(\vec{a})|$ and $V := \mathrm{vol}(C_r)$.

Here $\|\vec{x}\|_\square := \max\{|x_1|,\dots,|x_n|\}$.

So if $D\vec{G}$ doesn't change much on C_r, then

$$(1-\epsilon)^n |\det D\vec{G}(\vec{a})| \le \frac{\mathrm{vol}(\vec{G}(C_r))}{\mathrm{vol}(C_r)} \le (1+\epsilon)^n |\det D\vec{G}(\vec{a})|$$

is much as if we had applied a linear transformation.

Sketch of Theorem: Here is how $(*)$ follows from the Lemma. Enclose $\mathcal{J}$ in a big cube and subdivide it into little cubes C_i on which $D\vec{G}$ doesn't vary much. The Lemma gives

$$(1-\epsilon)^n |\det D\vec{G}(\vec{x}_i)| \operatorname{Vol}(C_i) \le \operatorname{Vol}(\vec{G}(C_i)) \le (1+\epsilon)^n |\det D\vec{G}(\vec{x}_i)| \operatorname{Vol}(C_i)$$

and letting m_i, M_i denote min/max values of $f \circ \vec{G}$ on C_i,

$$m_i \operatorname{Vol}(\vec{G}(C_i)) \le \int_{\vec{G}(C_i)} f\, dV \le M_i \operatorname{Vol}(\vec{G}(C_i)).$$

Combining these (and letting $\tilde{m}_i, \tilde{M}_i$ denote the min/max values of $(f \circ G) \cdot |\det D\vec{G}|$ on C_i) gives

$$(1-\epsilon)^n (\tilde{m}_i - \epsilon) \operatorname{vol}(C_i) \le \int_{\vec{G}(C_i)} f\, dV \le (1+\epsilon)^n (\tilde{M}_i + \epsilon) \operatorname{vol}(C_i).$$

Summing over i now gives

$$\sum_i \tilde{m}_i \operatorname{vol}(C_i) - \epsilon K \le \int_{\vec{G}(\mathcal{J})} f\, dV \le \sum_i \tilde{M}_i \operatorname{Vol}(C_i) + \epsilon K$$

where K is a constant not depending on the partition. Since we can take $\epsilon \to 0$ and the LHS/RHS are upper/lower Riemann sums for $\int_{\mathcal{J}} (f \circ G) |\det D\vec{G}|\, dV$, this is what the middle term must be. $\square$

The key to the proof of the formula

$$(*) \quad \boxed{\int_{G(\mathcal{J})} f(\vec{x}) \underbrace{dV}_{dx_1 \cdots dx_n} = \int_{\mathcal{J}} f(\vec{G}(\vec{u})) |\det D\vec{G}(\vec{u})| \underbrace{dV}_{du_1 \cdots du_n}.}$$

is the following:

open subset in $\mathbb{R}^n$

> **Lemma:** Suppose $\vec{G} : \mathcal{U} \to \mathbb{R}^n$ is C^1, $D\vec{G}$ is invertible, and $C_r \subset \mathcal{U}$ is a cube of width r centered at $\vec{a}$, on which
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> Then $\vec{G}(C_r)$ is a region, and
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> $$(1-\epsilon)^n \tau V \le \operatorname{vol}(\vec{G}(C_r)) \le (1+\epsilon)^n \tau V$$
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> where $\tau := |\det D\vec{G}(\vec{a})|$ and $V := \operatorname{vol}(C_r)$.

Here $\|\vec{x}\|_{\square} := \max\{|x_1|, \ldots, |x_n|\}$.

So if $D\vec{G}$ doesn't change much on C_r, then

$$(1-\epsilon)^n |\det D\vec{G}(\vec{a})| \le \frac{\operatorname{Vol}(\vec{G}(C_r))}{\operatorname{Vol}(C_r)} \le (1+\epsilon)^n |\det D\vec{G}(\vec{a})|$$

is much as if we had applied a linear transformation.

§3. What is a differential form?

These are the things we have been integrating all along! Not just the function f, but the whole expression after the integral sign. We'll discuss these more systematically from Thursday onward, but I will tell you what they are in $\mathbb{R}^3$ now in the simplest possible terms.

A few vector spaces

$\mathcal{A}^0 := \mathbb{R} =$ 1-dimensional vector space with basis $\{1\}$

$\mathcal{A}^1 :=$ 3-dimensional vector space with basis $\{\underbrace{dx, dy, dz}_{\text{for now just}}\}$

$\mathcal{A}^2 :=$ 3-dimensional vector space with basis $\{dx\wedge dy, dx\wedge dz, dy\wedge dz\}$

$\mathcal{A}^3 :=$ 1-dimensional vector space with basis $\{dx\wedge dy\wedge dz\}$

$$\begin{gathered} \| \\ -dy\wedge dx\wedge dz \\ \| \\ dy\wedge dz\wedge dx \\ \text{etc.} \end{gathered}$$

Differential forms $U \subset \mathbb{R}^3$ open set

A (smooth) differential k-form on U is a (C^∞, vector-valued) function from U to A^k, written

$$\omega \in A^k(U).$$

k is the degree of ω

k=0: A 0-form is a C^∞ function $f\begin{pmatrix}x\\y\\z\end{pmatrix}$.
That is, $A^0(U) = C^\infty(U)$.

k=1: A 1-form $\omega \in A^1(U)$ takes the form

$$\omega = P\begin{pmatrix}x\\y\\z\end{pmatrix}dx + Q\begin{pmatrix}x\\y\\z\end{pmatrix}dy + R\begin{pmatrix}x\\y\\z\end{pmatrix}dz.$$

k=2: A 2-form $\omega \in A^2(U)$ takes the form

$$\omega = P\begin{pmatrix}x\\y\\z\end{pmatrix}dy\wedge dz + Q\begin{pmatrix}x\\y\\z\end{pmatrix}dz\wedge dx + R\begin{pmatrix}x\\y\\z\end{pmatrix}dx\wedge dy.$$

k=3: A 3-form $\omega \in A^3(U)$ takes the form

$$\omega = f\begin{pmatrix}x\\y\\z\end{pmatrix}dx\wedge dy\wedge dz.$$

We will integrate 1-forms over curves, 2-forms over surfaces, etc.

Exterior derivative

If $f \in A^0(U)$ is a function, define a 1-form $df \in A^1(U)$ by

$$df := f_x\,dx + f_y\,dy + f_z\,dz.$$

We can extend this to higher degree, obtaining (for each k) maps

$$d : A^k(U) \longrightarrow A^{k+1}(U).$$

Here's how: require that d be zero on the basis vectors, $\mathbb{R}$-linear, and satisfy

$$d(f\omega) = df\wedge\omega + f\,d\omega.$$

For example

$$d(f\,dx) = f_x\overbrace{dx\wedge dx}^{0} + f_y\,dy\wedge dx + f_z\,dz\wedge dx$$
$$= -f_y\,dx\wedge dy - f_z\,dx\wedge dz.$$

As we'll see, this "d" nicely encapsulates "div", "grad", and "curl".