DIFFERENTIAL FORMS

22

§1. Differential forms in 3-space

VECTOR SPACES

$\mathcal{A}^0 := \mathbb{R}$ = 1-dimensional vector space with basis $\{1\}$

$\mathcal{A}^1 :=$ 3-dimensional vector space with basis $\{dx, dy, dz\}$

names of vectors

$\mathcal{A}^2$

$\mathcal{A}^3$

$\mathcal{A}^4$

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again, names of vectors.

$\mathcal{A}^3$

We will write

$dy \wedge dx = -dx \wedge dy$, etc.

$\mathcal{A}^4$

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$\mathcal{A}^3 :=$ 1-dimensional vector space with basis $\{dx \wedge dy \wedge dz\}$

$$\begin{aligned} &\| \\ &-dy \wedge dx \wedge dz \\ &\| \\ &dy \wedge dz \wedge dx \end{aligned}$$

etc.

$\mathcal{A}^4$

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$\mathcal{A}^4$ and higher are zero

N.B.: $dx\wedge dx = -dx\wedge dx \Rightarrow dx\wedge dx = 0$ etc.

DIFFERENTIAL FORMS

$U \subset \mathbb{R}^3$ open

A differential form of degree k (or k-form) on U, written $\omega \in \mathcal{A}^k(U)$, is a ($C^\infty$, vector-valued) function from U to $\mathcal{A}^k$.

($k=0$) A 0-form is a C^∞ function $f\begin{pmatrix}x\\y\\z\end{pmatrix}$. That is, $\mathcal{A}^0(U) = C^\infty(U)$.

($k=1$) A 1-form $\omega \in \mathcal{A}^1(U)$ takes the form

$$\omega = f\begin{pmatrix}x\\y\\z\end{pmatrix}dx + g\begin{pmatrix}x\\y\\z\end{pmatrix}dy + h\begin{pmatrix}x\\y\\z\end{pmatrix}dz.$$

($k=2$) A 2-form $\omega \in \mathcal{A}^2(U)$ is

$$\omega = f\begin{pmatrix}x\\y\\z\end{pmatrix}dy\wedge dz + g\begin{pmatrix}x\\y\\z\end{pmatrix}dx\wedge dz + h\begin{pmatrix}x\\y\\z\end{pmatrix}dx\wedge dy$$

($k=3$) A 3-form $\omega \in \mathcal{A}^3(U)$ looks like

$$\omega = f\begin{pmatrix}x\\y\\z\end{pmatrix}dx\wedge dy\wedge dz.$$

SUMS AND PRODUCTS

We can add forms of the same degree:

$$\left(\cancel{x\,dy\wedge dz} + y^2\,dx\wedge dy\right) + \left(z\,dz\wedge dx + \cancel{x\,\underbrace{dz\wedge dy}_{=-dy\wedge dz}}\right)$$

$$= y^2\,dx\wedge dy + z\,dz\wedge dx$$

We can take "wedge product" of forms of degrees k & l to get a form of degree $k+l$:

$$(x\,dy + y^2\,dz)\wedge(z\,dz + x^2\,dx)$$

$$= xz\,dy\wedge dz + \cancel{y^2 z\,\underbrace{dz\wedge dz}_{0}} + x^3\,dy\wedge dx + y^2x^2\,dz\wedge dx$$

Given forms $\omega\in\mathcal{A}^k(U)$ and $\eta\in\mathcal{A}^l(U)$,

$$\omega\wedge\eta = (-1)^{kl}\,\eta\wedge\omega.$$

Distributive and associative properties hold.

EXTERIOR DERIVATIVE

If $f\in\mathcal{A}^0(U)$ is a function (0-form), define a 1-form $df\in\mathcal{A}^1(U)$ by

$$df := \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

We can extend this to higher degree, obtaining (for each k) maps

$$d : \mathcal{A}^k(U)\to\mathcal{A}^{k+1}(U).$$

Here's how: require that d be zero on the basis vectors

$$1,\ dx,\ dy,\ dz,\ dx\wedge dy,\ dx\wedge dz,\ dy\wedge dz,\ dx\wedge dy\wedge dz$$

and that d be $\mathbb{R}$-linear

$\omega,\eta\in\Omega^k(U)$
$a,b\in\mathbb{R}$

$$d(a\omega + b\eta) = a\,d\omega + b\,d\eta$$

and satisfy the product rule

$$d(f\omega) = df\wedge\omega + f\,d\omega.$$

So for example

$$d(\underbrace{f\,dx}_{\text{1-form}}) = df \wedge dx + f \wedge \cancel{d(dx)}^{\,0}$$

$$= (f_x\,dx + f_y\,dy + f_z\,dz) \wedge dx$$

$$= f_x\,\cancel{dx \wedge dx}^{\,0} + f_y\,dy \wedge dx + f_z\,dz \wedge dx$$

$$= \underbrace{f_z\,dz \wedge dx - f_y\,dx \wedge dy}_{\text{2-form}}$$

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$$= (f_x\,dx + f_y\,dy + f_z\,dz) \wedge dx$$

$$= f_x\,\cancel{dx \wedge dx}^{\,0} + f_y\,dy \wedge dx + f_z\,dz \wedge dx$$

$$= f_z\,dz \wedge dx - f_y\,dx \wedge dy$$

and

$$d(\overbrace{f\,dx \wedge dy}^{\text{2-form}}) = df \wedge dx \wedge dy + f\,\cancel{d(dx \wedge dy)}^{\,0}$$

$$= f_x\,\underbrace{dx \wedge dx}_{0} \wedge dy + f_y\,\underbrace{dy \wedge dx}_{0} \wedge dy + f_z\,dz \wedge dx \wedge dy$$

$$= \underbrace{f_z\,dx \wedge dy \wedge dz}_{\text{3-form}}.$$

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$$= (f_x\,dx + f_y\,dy + f_z\,dz) \wedge dx$$

$$= f_x\,\cancel{dx \wedge dx}^{\,0} + f_y\,dy \wedge dx + f_z\,dz \wedge dx$$

$$= f_z\,dz \wedge dx - f_y\,dx \wedge dy$$

and

$$d(f\,dx \wedge dy) = df \wedge dx \wedge dy + f\,\cancel{d(dx \wedge dy)}^{\,0}$$

$$= f_x\,\underbrace{dx \wedge dx}_{0} \wedge dy + f_y\,\underbrace{dy \wedge dx}_{0} \wedge dy + f_z\,dz \wedge dx \wedge dy$$

$$= f_z\,dx \wedge dy \wedge dz.$$

(Clearly $d(f\,dx \wedge dy \wedge dz) = 0$.)

One can show that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{k}\,\omega \wedge d\eta$$

degree of ω

and

$$d(d\omega) = 0.$$

The second of these follows from the computation

$$d(df) = d(f_x\,dx + f_y\,dy + f_z\,dz)$$

$$= df_x \wedge dx + df_y \wedge dy + df_z \wedge dz$$

$$= (\cancel{f_{xx}\,dx} + f_{xy}\,dy + f_{xz}\,dz) \wedge dx$$

$$+ (f_{yx}\,dx + \cancel{f_{yy}\,dy} + f_{yz}\,dz) \wedge dy$$

$$+ (f_{zx}\,dx + f_{zy}\,dy + \cancel{f_{zz}\,dz}) \wedge dz$$

$$= (f_{yx} - f_{xy})\,dx \wedge dy + (f_{zx} - f_{xz})\,dx \wedge dz$$

$$+ (f_{zy} - f_{yz})\,dy \wedge dz$$

$= 0$ by Clairaut's Theorem.

($f_{xy} = f_{yx}$ etc.)

Problem: $d(f\,dx + g\,dy + h\,dz) =$

$$(f_y\,dy \wedge dx + f_z\,dz \wedge dx) + (g_x\,dx \wedge dy + g_z\,dz \wedge dy)$$

$$+ (h_x\,dx \wedge dz + h_y\,dy \wedge dz) =$$

$$(h_y - g_z)\,dy \wedge dz + (f_z - h_x)\,dz \wedge dx + (g_x - f_y)\,dx \wedge dy$$

curl?

$$d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) =$$

$$P_x\,dx \wedge dy \wedge dz + Q_y\,dy \wedge dz \wedge dx + R_z\,dz \wedge dx \wedge dy$$

$$= (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz.$$

div?

PULLBACKS

How do differential forms behave under smooth maps? Given ($V, U \subset \mathbb{R}^3$ open)

$$\vec{G} : \underset{u,v,w}{\overset{\text{coords.}}{V}} \longrightarrow \underset{x,y,z}{\overset{\text{coords}}{U}}$$

and a function (0-form) f on U,

$$\vec{G}^* f := f \circ \vec{G} \quad \text{"pullback of } f\text{"}$$

defines a function on V. Now writing $\vec{G}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} G_1\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ G_2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ G_3\begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{matrix}$ we define

$$\vec{G}^* dx := dG_1 = \frac{\partial G_1}{\partial u} du + \frac{\partial G_1}{\partial v} dv + \frac{\partial G_1}{\partial w} dw,$$

$$\vec{G}^* dy := dG_2, \quad \vec{G}^* dz := dG_3.$$

One then defines $\vec{G}^* \omega$ by applying $\vec{G}^*$ to every function and dx, dy, or dz in ω. This gives an $\mathbb{R}$-linear map

$$\vec{G}^* : A^k(U) \to A^k(V)$$
$$\omega \longmapsto G^* \omega$$

called the pullback.

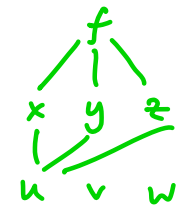
Let's try a computation:

$$\vec{G}^* df = \vec{G}^*(f_x dx + f_y dy + f_z dz)$$

$$= (f_x \circ \vec{G})\vec{G}^* dx + (f_y \circ \vec{G})\vec{G}^* dy + (f_z \circ \vec{G})G^* dz$$

$$= (f_x \circ \vec{G})\left(\frac{\partial G_1}{\partial u} du + \frac{\partial G_1}{\partial v} dv + \frac{\partial G_1}{\partial w} dw\right) + (f_y \circ \vec{G})\left(\frac{\partial G_2}{\partial u} du + \frac{\partial G_2}{\partial v} dv + \frac{\partial G_2}{\partial w} dw\right)$$

$$(f_z \circ \vec{G})\left(\frac{\partial G_3}{\partial u} du + \frac{\partial G_3}{\partial v} dv + \frac{\partial G_3}{\partial w} dw\right)$$



$$= \left((f_x \circ G)\frac{\partial G_1}{\partial u} + (f_y \circ G)\frac{\partial G_2}{\partial u} + (f_z \circ G)\frac{\partial G_3}{\partial u}\right) du$$

$$\left((f_x \circ G)\frac{\partial G_1}{\partial v} + (f_y \circ G)\frac{\partial G_2}{\partial v} + (f_z \circ G)\frac{\partial G_3}{\partial v}\right) dv + \left((f_x \circ G)\frac{\partial G_1}{\partial w} + (f_y \circ G)\frac{\partial G_2}{\partial w} + (f_z \circ G)\frac{\partial G_3}{\partial w}\right) dw$$

$$= \frac{\partial (f \circ \vec{G})}{\partial u} du + \frac{\partial (f \circ \vec{G})}{\partial v} dv + \frac{\partial (f \circ \vec{G})}{\partial w} dw$$

$$= d(f \circ \vec{G}) = d(\vec{G}^* f).$$

So pullback and exterior derivative commute on f (and on dx, dy, dz), hence on all forms:

$$d(\vec{G}^* \omega) = \vec{G}^*(d\omega).$$

PULLBACKS

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One then defines $\vec{G}^* \omega$ by applying $\vec{G}^*$ to every function and dx, dy, or dz in ω. This gives an $\mathbb{R}$-linear map

$$\vec{G}^* : \mathcal{A}^k(U) \to \mathcal{A}^k(V)$$
$$\omega \longmapsto G^* \omega$$

Called the <u>pullback</u>.

<u>Problem</u>: Writing $\vec{G}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ y\begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ z\begin{pmatrix} u \\ v \\ w \end{pmatrix} \end{pmatrix}$,

$$\vec{G}^*(dx \wedge dy) = (x_u du + x_v dv + x_w dw) \wedge (y_u du + y_v dv + y_w dw)$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du \wedge dv + \begin{vmatrix} x_v & x_w \\ y_v & y_w \end{vmatrix} dv \wedge dw + \begin{vmatrix} x_u & x_w \\ y_u & y_w \end{vmatrix} du \wedge dw$$

$$\vec{G}^*(dx \wedge dy \wedge dz) = (x_u du + x_v dv + x_w dw) \wedge (y_u du + y_v dv + y_w dw) \wedge (z_u du + z_v dv + z_w dw)$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} du \wedge dv \wedge dw = (\det D\vec{G})\, du \wedge dv \wedge dw.$$

The upshot is that there is a relation between pullbacks of top-degree forms and the Jacobian determinant which enters in the change-of-variable formula.

§2. Integrating differential forms

Let $S \subset \mathbb{R}^3$ be a region, and U an open set containing it on which a function $f \in C^\infty(U)$ is given.

Definition: The integral of the 3-form $\omega := f\, dx \wedge dy \wedge dz$ over S is

$$\int_S \omega := \int_S f\, dV.$$

Suppose a C^∞ map $\vec{G} : V \to U$ is given, which maps a region $T \subset V$ in 1-to-1 fashion to S $(= \vec{G}(T))$.

By the change-of-variables theorem,

$$\int_S \underbrace{f\, dV}_{\omega} = \int_T \underbrace{(f \circ \vec{G})\, |\det D\vec{G}|\, dV}_{\vec{G}^* \omega}.$$

By the last "problem", this translates to:

Proposition: Assuming $\det D\vec{G} > 0$,

$$\int_{\vec{G}(T)} \omega = \int_T \vec{G}^* \omega$$

for any $\omega \in A^3(U)$.

This works in any dimension: replacing $U \subset \mathbb{R}^3$ by $U \subset \mathbb{R}^m$, take $\omega \in A^m(U)$; then the Proposition holds. Why do we care? Because it allows us to define integration on manifolds.

Namely, suppose $\vec{g} : U \to \mathbb{R}^n$ is $C^\infty/1\text{-}1$, $\vec{g}(S) = M$ is a manifold, and let $\eta \in A^m(\mathbb{R}^n)$. Then we set

$$\int_M \eta := \int_S \vec{g}^* \eta.$$

This will make sense if it is independent of how we parametrized M.

So consider $\vec{h} : V \to \mathbb{R}^n$, also C^∞/1-1 with $\vec{h}(\sigma) = M$;

open set in $\mathbb{R}^m$

region in V

and set $\vec{G} := \vec{g}^{-1} \circ \vec{h} : \sigma \to \mathcal{S}$.

Then $\vec{h} = \vec{g} \circ (\vec{g}^{-1} \circ \vec{h})$

$$\Rightarrow \vec{h}^* = (\vec{g}^{-1} \circ \vec{h})^* \circ \vec{g}^* = \vec{G}^* \circ \vec{g}^*$$

$$\Rightarrow \int_\sigma \vec{h}^* \eta = \int_\sigma \vec{G}^*(\vec{g}^* \eta) = \int_{\mathcal{S}} \vec{g}^* \eta$$

by the Proposition.

Now going back to 3-space, we have the opportunity to integrate

- 3-forms over regions ✓
- 2-forms over surfaces (later)
- 1-forms over curves:

Let $\omega = P\begin{pmatrix}x\\y\\z\end{pmatrix}dx + Q\begin{pmatrix}x\\y\\z\end{pmatrix}dy + R\begin{pmatrix}x\\y\\z\end{pmatrix}dz$, and $\vec{g} : [a,b] \to \mathbb{R}^3$ parametrize a curve C in 1-1 fashion (with $\vec{g}' \neq 0$).

Writing $\vec{g}(t) = \begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix}$, we have by definition

$$\int_C \omega = \int_{[a,b]} \vec{g}^* \omega$$

$$= \int_a^b P(\vec{g}(t))\overbrace{dx(t)}^{x'(t)\,dt} + Q(\vec{g}(t))\overbrace{dy(t)}^{y'(t)\,dt} + R(\vec{g}(t))\underbrace{dz(t)}_{z'(t)\,dt}$$

$\vec{F} := \begin{pmatrix}P\\Q\\R\end{pmatrix}$ vector field

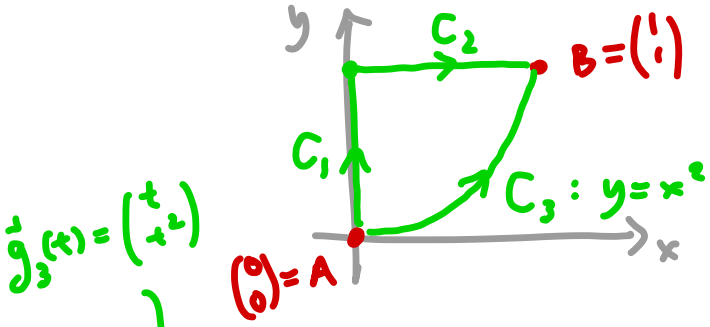
$$= \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\, dt$$

$$= \int_a^b \vec{F}(\vec{g}(t)) \cdot \frac{\vec{g}'(t)}{\|\vec{g}'(t)\|} \|\vec{g}'(t)\|\, dt$$

$$=: \text{"}\int_C \vec{F} \cdot \vec{T}\, ds\text{"}.$$

So we recover the standard line integral, which computes work if $\vec{F}$ is a force field.

Ex/ Integrate $\omega = xy^2dx + (x^2-y^2)dy$ over the two paths from A to B shown:



$$\int_{C_3}\omega = \int_{[0,1]} g_3^*\left(xy^2dx + (x^2-y^2)dy\right)$$

$$= \int_0^1 \{x(t)\,y(t)^2\,\underbrace{x'(t)}_{1} + (x(t)^2-y(t)^2)\,\underbrace{y'(t)}_{2t}\}\,dt$$

$$= \int_0^1 \{\cancel{t^5} + 2t^3 - \cancel{2}t^5\}\,dt = \frac{1}{3}.$$

$$\int_{C_1+C_2}\omega = \int_{C_1}\omega + \int_{C_2}\omega = \int_{[0,1]} g_1^*\omega + \int_{[0,1]} g_2^*\omega$$

$\vec{g}_1(t)=\binom{0}{t}$, $\vec{g}_1^*dx=0$

$\vec{g}_2(t)=\binom{t}{1}$, $\vec{g}_2^*dy=0$

$$= \int_0^1 (0-t^2)\,dt + \int_0^1 t(1)^2\,dt = -\frac{1}{3}+\frac{1}{2}=\frac{1}{6}.$$

Conclude that the integral of ω depends on the path from A to B!

FINAL NOTE

I did not cover Shifrin's "multilinear setup". Please look this over (§8.2.1).

The basic idea is to interpret, say, $dx\wedge dz$ as an alternating multilinear function on pairs of vectors in $\mathbb{R}^3$:

given $\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ & $\vec{w}=\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$,

$$dx\wedge dz\,(\vec{v},\vec{w}) := \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix}$$ is

the area of the parallelogram spanned by the projections of $\vec{v}$ & $\vec{w}$ to the xz-plane.

The notation for alternating multilinear functions on k-tuples of vectors in $\mathbb{R}^n$ is $\Lambda^k(\mathbb{R}^n)^*$; it is spanned by the $\{dx_{i_1}\wedge\cdots\wedge dx_{i_k}\}_{i_1<\cdots<i_k}$ and has dimension $\binom{n}{k}$.