GREEN'S THEOREM 23
According to Newton, the magnitude of force of gravitational attraction between objects of masses $M$ and $m$ is $\frac{GMm}{d^2}$, where $G$ is a universal constant and $d$ is the distance between the objects. According to Ptolemy, the Earth $(=M)$ sits at the origin. Let's describe the force field it exerts on an object of mass $m$: writing $\vec{r} = (\frac{x}{y})$ for its position and $r = ||\vec{r}||$ for its distance from $O$, the force has magnitude

$$||F(\vec{r})|| = \frac{GMm}{r^2} = \frac{GMm}{||\vec{r}||^3}.$$

The direction is toward the origin, i.e. in the direction of the unit vector $\frac{\vec{r}}{||\vec{r}||}$.

$\Rightarrow \vec{F}(\vec{r}) = \frac{-GMm}{||\vec{r}||^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-GMm}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

Notice that

$$\frac{\partial}{\partial x} \frac{1}{||\vec{r}||} = \frac{\partial}{\partial x} \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{11x11^3}$$

and similarly $\frac{\partial}{\partial y} \frac{1}{||\vec{r}||} = \frac{-y}{11y11^3}$

$\frac{\partial}{\partial z} \frac{1}{||\vec{r}||} = \frac{-z}{11z11^3}$, so that

$$\nabla \frac{GMm}{||\vec{r}||} = F(\vec{r}).$$

Writing $\varphi(\vec{r}) := \frac{GMm}{||\vec{r}||}$, we say that $\varphi$ is a potential function for $F$, and define the potential energy of the object moving in the force field to be

$$PE := -\varphi(\vec{r}).$$

(This is well-defined up to a constant.)

The 1-form $\omega = -GMm \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$

corresponds to $F$, and $d\varphi = \omega.$
Recall that the work done by $\vec{F}$ as the object moves along a curve $C$ is

$$W = \Delta KE = \int_C \omega$$

(parametrized by $\vec{r} : [a, b] \to \mathbb{R}^3$) (or $\int_C \vec{F} \cdot d\vec{r}$)

It would be nice if this was (up to a constant)

$$-\Delta PE = \varphi(B) - \varphi(A) = GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right),$$

so that energy is conserved.

This is precisely the content of today's first main theorem. Accordingly, we will call a vector field conservative if it is the gradient of a function.
Proof: \( r^* df = d (r^* f) = d (f o r) \)
\[ = (f o r)' (t) \, dt \]
\[ \Rightarrow \int_C df = \int_a^b r^* df = \int_a^b (f o r)' (t) \, dt \]
\[ = f(r (t)) \bigg|_a^b = f(B) - f(A). \]

If \( C \) is only piecewise smooth, then
\( A = A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n = B \)
\[ \int_C df = \sum_{i=1}^n \int_{C_i} df = \sum_{i=1}^n (f(A_i) - f(A_{i-1})) \]
\[ = f(B) - f(A). \]

\[ \quad \square \]

Let \( \omega = \sum_{i=1}^n F_i (r^i) \, dx_i \in A^1 (U) \) be an arbitrary 1-form, with corresponding vector field \( \vec{F} (r^i) = \begin{pmatrix} F_1 (r^1) \\ \vdots \\ F_n (r^n) \end{pmatrix} \).

\[ \int_C \omega = \int_C \sum_{i=1}^n F_i (r^i) \, dx_i = \sum_{i=1}^n \int_C F_i (r^i) \, dx_i. \]

\[ \text{Theorem: } \int_C \omega = f(B) - f(A) \]

or \( \int_C df \), or \( \int_C \nabla f \cdot dr \)
Proof: \((3) \Rightarrow (1)\) is the FTC.

\((1) \Rightarrow (2)\)

Given \(\hat{A} \rightarrow C \rightarrow \hat{B}\)

the curve \(\hat{A} \rightarrow C \rightarrow C' \rightarrow \hat{B}\) is closed,

and \(\int_C \omega - \int_{C'} \omega = \phi \omega = 0\).

\((2) \Rightarrow (3)\)

Defining \(f(x) := \int_{\hat{A}}^x \omega\),

\[
\frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x + h \hat{\varepsilon}) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x + h \hat{\varepsilon}} \omega
\]

\[
= \lim_{h \to 0} \frac{1}{h} \int_0^h F_i(x + \hat{\varepsilon}_t) \, dt = F_i(x).\]

\(f(x) + \varepsilon \Rightarrow dx \neq d\varepsilon\)

\[\Rightarrow df = \omega.\]

We shall say that "\(\int \omega\) is independent of path" if for each \(\hat{A}, \hat{B} \in U\), any curve \(C\) starting at \(\hat{A}\) and ending at \(\hat{B}\) yields the same value \(\int_C \omega\).

A curve \(C\) is closed if it starts \(\hat{A}\) and ends at the same point, in which case we write \(\int_C \omega\) for the line integral.

Proposition: TFAE

(1) \(\int_C \vec{F} \cdot d\vec{r} = 0\) \(\forall\) closed \(C\)

(2) \(\int_{\hat{A}}^{\hat{B}} \vec{F} \cdot d\vec{r}\) is independent of path

(3) \(\vec{F}\) is conservative (= \(\nabla f\))
§2. Finding a potential function

How can we tell if a 1-form $\omega$ is exact (i.e. if $\bar{F}$ is conservative)?

Well, if $\omega = df$, then

$$dw = d(df) = 0,$$

i.e. $\omega$ is closed. This is easy to check, so might it be the case that closed $\Rightarrow$ exact?

**Problem:** Which of the following are closed? Which do you think are exact on their domains?

(i) $\omega = (4x^3 + 9xy^2)\,dx + (6x^2y + 6y^5)\,dy$

(ii) $\omega = xy^2\,dx + (x^2 - y^2)\,dy$

(iii) $\omega = \frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy$

We shall say that "$\omega$ is independent of path" if for each $\vec{A}, \vec{B} \in U$, any curve $C$ starting at $\vec{A}$ and ending at $\vec{B}$ yields the same value $\int_C \omega$.

![Diagram](attachment:diagram.png)

A curve $C$ is closed if it starts and ends at the same point, in which case we write $\int_C \omega$ for the line integral.

**Proposition:** TFAE

(1) $\int_C \bar{F} \cdot d\vec{x} = 0 \forall$ closed $C$

(2) $\int_A^B \bar{F} \cdot d\vec{x}$ is independent of path

(3) $\bar{F}$ is conservative ($= \bar{\nabla} f$)

$\omega$ is exact ($= df$)
Problem: Try this (for (ii)).

\[ f_x(x, y) = 4x^3 + 9x^2 y^2 \quad \Rightarrow \quad 2y \]
\[ f(x, y) = x^4 + 3x^2 y^3 + h(y) \quad \Rightarrow \quad 6x^3 y + 6y^5 = f_y = 0 + 6x^2 y + h'(y) \]
\[ h'(y) = 6y^5 \quad \Rightarrow \quad h(y) = y^6 + C \quad \Rightarrow \quad f(x, y) = x^4 + 3x^2 y^3 + y^6 + C. \] (so \(w\) is exact)

Finally, for \(w = \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy\), consider the integral around \(C\):

parametrize by \(\vec{r}(t) = (\cos t, \sin t)\)
gives \(\oint_C w = \int_0^{2\pi} \vec{r}'(t) \cdot w \, dt\)
\[ = \int_0^{2\pi} -\sin t \, (\cos^2(t)) + \cos t \, (\sin^2(t)) \, dt \]
\[ = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt = 2\pi \neq 0 \]
so by the Proposition \(w\) CANNOT be exact. We'll come back to this.

First write
\[ d(P \, dx + Q \, dy) = P_y \, dy \, dx + Q_x \, dx \, dy \]
\[ = (Q_x - P_y) \, dx \, dy. \]

(i) \(Q_x - P_y = \frac{\partial}{\partial x}(6x^3 y + 6y^5) - \frac{\partial}{\partial y}(4x^3 + 9x^2 y^2)\)
\[ = 18x^2 y - 18x^2 y = 0 \]

(ii) \(Q_x - P_y = \frac{\partial}{\partial x}(x^2 - y^4) - \frac{\partial}{\partial y}(xy^2)\)
\[ = 2x - 2xy \neq 0 \]

(iii) \(Q_x - P_y = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{y}{x^2+y^2})\)
\[ = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = 0 \]

We already know exact \(\Rightarrow\) closed, so not closed \(\Rightarrow\) not exact. Hence (ii) is not exact.

For (i), we can suppose that \(w\) is exact, \(w = df = f_x \, dx + f_y \, dy\), and try to solve \(f_x = P, f_y = Q\).
2. Finding a potential function

How can we tell if a 1-form $\omega$ is exact (i.e. if $\Phi$ is conservative)?

Well, if $\omega = df$, then

$$\text{d} \omega = \text{d}(\text{d} f) = 0,$$

i.e. $\omega$ is closed. This is easy to check, so might it be the case that $\text{closed} \Rightarrow \text{exact}$?

Suppose that $\omega \in \Omega^1(U)$, with $U$ star-shaped:

that is, for some point $\hat{A} \in U$, every point of $U$ is at the end of a line segment starting at $\hat{A}$.

I claim that in this case $(\star)$ holds:

$$0 = \text{d} \omega = \text{d} \sum_{i,j} F_{ij} \text{d} x_i \wedge \text{d} x_j = \sum_{i,j} \frac{\partial F_{ij}}{\partial x_i} \text{d} x_i \wedge \text{d} x_j$$

$$= \sum_{i,j} \left( \frac{\partial F_{ij}}{\partial x_i} - \frac{\partial F_{ij}}{\partial x_j} \right) \text{d} x_i \wedge \text{d} x_j.$$

forcing $\frac{\partial F_{ij}}{\partial x_i} = \frac{\partial F_{ij}}{\partial x_j}$ for all $i,j$.

For notational simplicity assume $\hat{A} = \emptyset$, and put

$$f(\hat{x}) := \int_{\hat{x}}^{\hat{x}'} \omega = \int_{0}^{1} \hat{x}^* \omega$$

$$= \int_{0}^{1} \sum F_{i}(\hat{x}) \text{d} x_i$$

$$(\text{d} x_i = x_i \text{d} t)$$

$$= \sum_{i,j} x_j \int_{0}^{1} F_{j}(\hat{x}) \text{d} t$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(\hat{x}) = \int_{0}^{1} F_{i}(\hat{x}') \text{d} t + \sum_{j} x_{j} \int_{0}^{1} t \frac{\partial F_{i}}{\partial x_{j}}(\hat{x}') \text{d} t$$

$$= \int_{0}^{1} F_{i}(\hat{x}') \text{d} t + \int_{0}^{1} t \sum_{j} x_{j} \frac{\partial F_{i}}{\partial x_{j}}(\hat{x}') \text{d} t$$

$$\text{by parts}$$

$$= t F_{i}(\hat{x}') \bigg|_{0}^{1} \quad \frac{\partial F_{i}}{\partial x_{i}}(\text{chain rule})$$

$$= F_{i}(\hat{x}).$$

So $\text{d} f = \sum F_{i}(\hat{x}) \text{d} x_i = \omega.$
The most general result is that we A'(U) closed (\( dw = 0 \)) \( \Rightarrow \) exact (\( \omega = df \)) as long as \( U \) is simply connected, i.e., has no holes.

I claim that in this case \((\ast)\) holds:

\[
0 = dw = d \sum_{j} F_{j} \, dx_{j} = \sum_{ij} \frac{\partial F_{i}}{\partial x_{j}} \, dx_{j} \wedge dx_{i} = \sum_{ij} \left( \frac{\partial F_{i}}{\partial x_{i}} - \frac{\partial F_{i}}{\partial x_{j}} \right) \, dx_{i} \wedge dx_{j},
\]

for \( \frac{\partial F_{j}}{\partial x_{i}} = \frac{\partial F_{i}}{\partial x_{j}} \) for all \( i, j \).

For notational simplicity assume \( A = 0 \), and put

\[
f(t_{\omega}) = \int_{0}^{t_{\omega}} \omega = \int_{0}^{t_{\omega}} \omega = \int_{0}^{t_{\omega}} \left[ \sum_{j} \frac{\partial F_{i}}{\partial x_{j}} \right] \, dx_{j} = \int_{0}^{t_{\omega}} \sum_{j} F_{j} \left( t_{\omega} x_{j} \right) \, dx_{j} = \sum_{j} x_{j} \int_{0}^{t_{\omega}} F_{j} \left( t_{\omega} x_{j} \right) \, dx_{j} = \frac{\partial f}{\partial x_{i}} (t_{\omega}) = \frac{\partial f}{\partial x_{i}} (t_{\omega}) = \left[ \frac{\partial F_{i}}{\partial x_{i}} (t_{\omega} x_{j}) \right]_{0}^{1} = \left[ \frac{\partial F_{i}}{\partial x_{i}} (t_{\omega} x_{j}) \right]_{0}^{1} = F_{i}(t_{\omega}).
\]

So \( df = \sum F_{i}(t_{\omega}) \, dx_{i} = \omega \).
The most general result is that \( \omega \wedge A(w) \) closed \( (d\omega = 0) \) \Rightarrow \text{exact} \ (\omega = df) \) as long as \( U \) is simply connected, i.e. has no holes:

\[
\text{bad}
\]

This is precisely what goes wrong with \( \omega = \frac{-y}{x^2+y^2} \ dx + \frac{x}{x^2+y^2} \ dy \) : it lines on \( \mathbb{R}^2 \setminus \{0\} \), which is not simply connected. Locally it is \( d(\arctan(y/x)) \), but \( \Theta \) does not yield a well-defined function globally on \( \mathbb{R}^2 \setminus \{0\} \) (going around the origin, it changes by \( 2\pi \)).

\[\text{Ex/ Let } C \text{ be some path in } \mathbb{R}^3 \text{ from } \hat{A} = (0,0,1) \text{ to } \hat{B} = (1,0,-2), \text{ and } \]
\[
\omega = \left( e^{x \cos y + y^2} \right) dx + \left( x e^{x \sin y} \right) dy + (xy + 2z) dz.
\]

Find \( \int_C \omega \).

I’ll just outline the solution:

- Check \( d\omega = 0 \).
- Since \( \omega \) is defined on \( \mathbb{R}^3 \), which is certainly star-shaped, this implies \( \omega = df \).
- Find \( f \) by setting \( f_x = P, f_y = Q, f_z = R \). We get \( f = e^{x \cos y + xy^2 + z} \).
- Apply the FTC:

\[
\int_C \omega = f(B) - f(A) = (9 - \frac{3\pi}{2}) - 3 = 6 - \frac{3\pi}{2}.
\]
3. Green's Theorem (in $\mathbb{R}^2$)

This is an analogue of the FTC

$$\int_C df = \int_{\partial C} f$$ (i.e. $f(B) - f(A)$)

for integrals of 2-forms over a region.

Suppose $\omega = P(x,y)dx + Q(x,y)dy$ is any 1-form defined on an open set $U \subset \mathbb{R}^2$, and $\partial \mathcal{B} \subset U$ is a region with boundary curve $\partial \mathcal{B}$ piecewise $C^1$.

$\partial \mathcal{B}$ is oriented so that $\partial \mathcal{B}$ is "on the left" (counterclockwise orientation).

**Theorem:** $\oint_{\partial \mathcal{B}} \omega = \int_{\mathcal{B}} d\omega$.

**Notes:**

- This is in the same spirit as the FTC, in that it uses boundary values of $P$ & $Q$ to compute the integral of $d\omega = (Q_x - P_y) dx \wedge dy$ only on the interior.

- It implies the "closed $\Rightarrow$ exact" result for $U$ simply connected, which we can define to mean that the region enclosed by a simple closed curve $C$ is contained in $U$. Then $\omega$ closed ($d\omega = 0$) $\Rightarrow \oint_{\partial \mathcal{B}} \omega = 0$ $\Rightarrow \int_{\mathcal{B}} d\omega = 0$ $\Rightarrow \omega$ exact.
Proof for rectangle:

\[ \int_{\partial R} \omega = \sum_{i} \int_{c_i} \omega = \int_{c_1} \omega + \int_{c_2} \omega + \int_{c_3} \omega + \int_{c_4} \omega = \int_{a}^{b} P(x) dx + \int_{c}^{d} Q(y) dy - \int_{a}^{b} P(x) dx - \int_{c}^{d} Q(y) dy \]

On the other side,

\[ \int_{R} d\omega = \int_{R} (Q_y - P_x) dx dy \]

\[ = \int_{c}^{d} \int_{a}^{b} Q_y dx dy - \int_{a}^{b} \int_{c}^{d} P_x dy dx \]

\[ = \int_{c}^{d} Q_y dy - \int_{c}^{d} Q(y) dy \]

\[ - \left( \int_{a}^{b} P(x) dx - \int_{a}^{b} P(x) dx \right) \]

which equals the expression for \( \int_{\partial R} \omega \).

Theorem: \( \int_{R} d\omega = \int_{\partial R} \omega \)

Proof for general region: Break \( R \) up into little subregions parametrized by rectangles and observe their internal boundaries cancel so that \( \sum d\varepsilon_{ij} = d\varepsilon \). So it suffices to prove for \( R \) the image of a 1-1 C' map \( \tilde{G}: R \to \hat{R} \). But

\[ \int_{\hat{R}} \omega = \int_{\hat{R}} \tilde{G}^{*} \omega = \int_{R} d \tilde{G}^{*} \omega \]

\( \tilde{G}(R) \) drawn for rectangle

\[ = \int_{R} \tilde{G}^{*}(d\omega) = \int_{R} d\omega \]

\( \tilde{G}(R) \).
Ex. Find the area under one arch of the cycloid parametrized by
\[ \vec{r}(t) = (x(t), y(t)) = (t - \sin(t), 1 - \cos(t)). \]
(t ∈ [0, 2π])

If w = x dy, then dw = dx dy.
So Area(S) = \( \int_S dw = \int_S x dy \)
= \( -\int_{C_1} x dy + \int_{C_2} x dy = -\int_{C_2} x dy \)
= \( -\int_0^{2\pi} x(t) y'(t) dt = \int_0^{2\pi} (-\sin(t) - t) \sin(t) dt \)
= \( \int_0^{2\pi} \left( \frac{1 - \cos 2t}{2} - t \sin t \right) dt \)
= \( \left[ \frac{t}{2} - \frac{1}{4} \sin 2t - \sin t + t \cos t \right]_0^{2\pi} \)
= \( 3\pi \).

\[
\text{Theorem: } \int_S dw = \oint_C w
\]

Ex. Find the line integral \( \oint_C \left( -y dx + x dy \right) \)
\( \vec{c} \)

\( ds = \sqrt{x'^2 + y'^2} \)

and we know that \( \oint_{C'} w = 2\pi \).

We also know \( dw = 0 \),
so
\( 0 = \int_S dw = \int_{C'} w = \int_C w - \int_C w \)
= \( \int_C w - 2\pi \)
\( \Rightarrow \oint_C w = 2\pi \).