Green's Theorem 23

According to Newton, the magnitude of force of gravitational attraction between objects of masses M & m is $\frac{GMm}{d^2}$, where G is a universal constant and d is the distance between the objects.

According to Ptolemy, the Earth ($=M$) sits at the origin. Let's describe the force field it exerts on an object of mass m: writing $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for its position and $r = \|\vec{x}\|$ for its distance from $\vec{0}$, the force has magnitude

$$\|\vec{F}(\vec{x})\| = \frac{GMm}{r^2} = \frac{GMm\|\vec{x}\|}{\|\vec{x}\|^3}.$$

The direction is toward the origin, i.e. in the direction of the unit vector $\frac{-\vec{x}}{\|\vec{x}\|}$

$$\Rightarrow \vec{F}(\vec{x}) = \frac{-GMm\,\vec{x}}{\|\vec{x}\|^3} = -GMm\,\frac{x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3}{(x^2+y^2+z^2)^{3/2}}$$

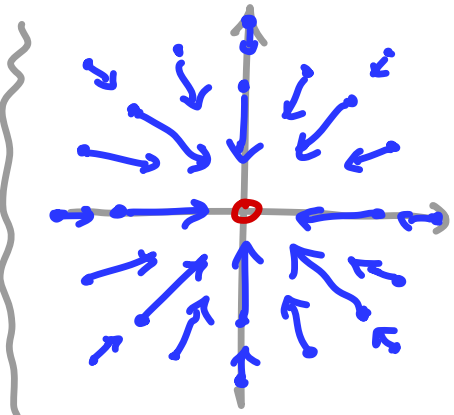
Notice that

$$\frac{\partial}{\partial x}\frac{1}{\|\vec{x}\|} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{\|\vec{x}\|^3}$$

and similarly $\frac{\partial}{\partial y}\frac{1}{\|\vec{x}\|} = \frac{-y}{\|\vec{x}\|^3}$

& $\frac{\partial}{\partial z}\frac{1}{\|\vec{x}\|} = \frac{-z}{\|\vec{x}\|^3}$, so that



2-D cross section

$$\vec{\nabla}\frac{GMm}{\|\vec{x}\|} = \vec{F}(\vec{x}).$$

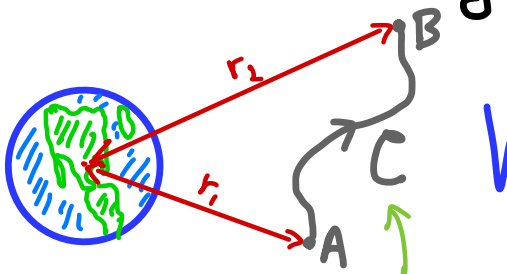
Writing $\varphi(\vec{x}) := \frac{GMm}{\|\vec{x}\|}$, we say that φ is a <u>potential function</u> for $\vec{F}$, and define the <u>potential energy</u> of the object moving in the force field to be

$$PE := -\varphi(\vec{x}).$$

(This is well-defined up to a constant.)

The 1-form $\omega = -GMm\,\frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$ corresponds to $\vec{F}$, and $d\varphi = \omega$.

Recall that the work done by $\vec{F}$ as the object moves along a curve C is



$$W = \Delta KE = \underbrace{\int_C \omega}_{\left(\text{or } \int_C \vec{F}\cdot d\vec{r}\right)}$$

C (parametrized by $\vec{r} : [a,b] \to \mathbb{R}^3$)

It would be nice if this was (up to a constant)

$$-\Delta PE = \varphi(B) - \varphi(A)$$
$$= GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right),$$

so that energy is conserved.

This is precisely the content of today's first main theorem. Accordingly, we will call a vector field conservative if it is the gradient of a function.

§1. The FTC for line integrals

Proof: $\vec{r}^{\,*} df = d(\vec{r}^{\,*} f) = d(f \circ \vec{r})$

$$= (f \circ \vec{r})'(t)\, dt$$

C smooth

$$\Rightarrow \int_C df = \int_a^b r^* df = \int_a^b (f \circ \vec{r})'(t)\, dt$$

$$= f(\vec{r}(t))\Big|_a^b = f(\vec{B}) - f(\vec{A}).$$

If C is only piecewise smooth, then

$$\int_C df = \sum_{i=1}^n \int_{C_i} df = \sum_{i=1}^n \left(f(\vec{A}_i) - f(\vec{A}_{i-1})\right)$$

$$= f(\vec{B}) - f(\vec{A}). \qquad \square$$

telescoping sum

Let $\omega = \sum_{i=1}^n F_i(\vec{x})\, dx_i \in \mathcal{A}^1(U)$ be an arbitrary 1-form, with corresponding vector field $\vec{F}(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_n(\vec{x}) \end{pmatrix}$.

§1. The FTC for line integrals

- Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}$ a C^1 function, with $\omega = df \left(= \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) \in \mathcal{A}^1(U)$. Such a 1-form is called <u>exact</u>.
- Let $C \subset U$ be a piecewise smooth curve parametrized by $\vec{r} : [a,b] \to \mathbb{R}^n$, with $\vec{A} = \vec{r}(a)$ & $\vec{B} = \vec{r}(b)$.

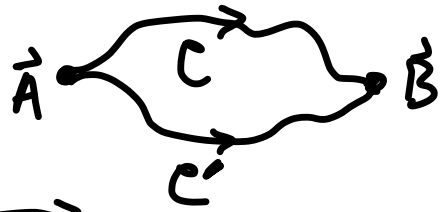
oriented curve

$$\boxed{\textbf{Theorem: } \int_C \omega = f(\vec{B}) - f(\vec{A})}$$

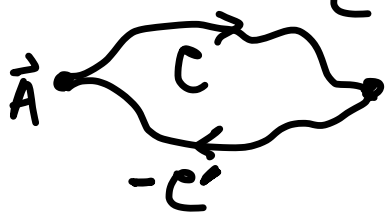
or $\int_C df$, or $\int_C \vec{\nabla} f \cdot d\vec{r}$

Proof: $\boxed{(3) \Rightarrow (1)}$ is the FTC.

$\boxed{(1) \Rightarrow (2)}$ Given



the curve



is closed,

and $\displaystyle\int_C \omega - \int_{C'} \omega = \oint_{C-C'} \omega = 0.$

$\boxed{(2) \Rightarrow (3)}$ Defining $f(\vec{x}) := \displaystyle\int_{\vec{A}}^{\vec{x}} \omega$, ← fixed point

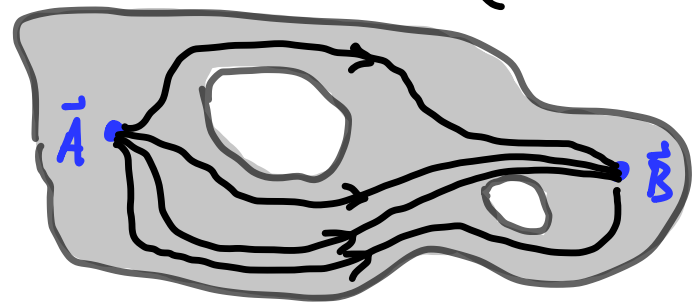
$$\frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x}+h\hat{e}_i) - f(\vec{x})}{h} = \lim_{h\to 0} \frac{1}{h}\int_{\vec{x}}^{\vec{x}+h\hat{e}_i} \omega$$

$$= \lim_{h\to 0} \frac{1}{h}\int_0^h \underbrace{F_i(\vec{x}+t\hat{e}_i)\,dt}_{\vec{r}^*\omega} = F_i(\vec{x}).$$

$\vec{r}(t) = \vec{x} + t\hat{e}_i$
$\Rightarrow dx_i = dt$

$$\Rightarrow df = \omega. \qquad \square$$

We shall say that "$\int \omega$ is independent of path" if for each $\vec{A}, \vec{B} \in U$, any curve C starting at $\vec{A}$ and ending at $\vec{B}$ yields the same value $\int_C \omega$.



A curve C is closed if it starts & ends at the same point, in which case we write $\oint_C \omega$ for the line integral.

Proposition: TFAE

(1) $\oint_C \vec{F}\cdot d\vec{r} = 0 \quad \forall$ closed C

(2) $\displaystyle\int_A^B \vec{F}\cdot d\vec{r}$ is independent of path

(3) $\vec{F}$ is conservative $(= \vec{\nabla} f)$

§2. Finding a potential function

How can we tell if a 1-form ω is exact (i.e. if $\vec{F}$ is conservative)?

Well, if $\omega = df$, then

$$d\omega = d(df) = 0,$$

i.e. ω is closed. This is easy to check, so might it be the case that closed $\Rightarrow$ exact?

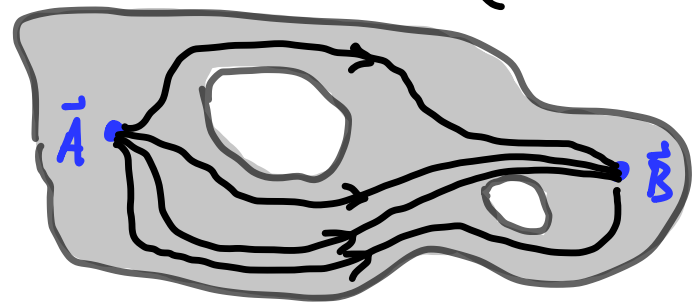
PROBLEM: Which of the following are closed? Which do you think are exact on their domains?

(i) $\omega = (4x^3 + 9x^2y^2)\,dx + (6x^3y + 6y^5)\,dy$

(ii) $\omega = xy^2\,dx + (x^2 - y^2)\,dy$

(iii) $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

We shall say that "$\int \omega$ is independent of path" if for each $\vec{A}, \vec{B} \in U$, any curve $\mathcal{C}$ starting at $\vec{A}$ and ending at $\vec{B}$ yields the same value $\int_{\mathcal{C}} \omega$.



A curve $\mathcal{C}$ is closed if it starts & ends at the same point, in which case we write $\oint_{\mathcal{C}} \omega$ for the line integral.

Proposition: TFAE

(1) $\oint_{\mathcal{C}} \underset{\omega}{\vec{F}\cdot d\vec{r}} = 0 \quad \forall$ closed $\mathcal{C}$

(2) $\int_A^B \underset{\omega}{\vec{F}\cdot d\vec{r}}$ is independent of path

(3) $\vec{F}$ is conservative $(= \vec{\nabla} f)$
ω is exact $(= df)$

PROBLEM: Try this (for (i)).

$$f_x(x,y) = 4x^3 + 9x^2y^2 \xrightarrow[\text{in } x]{\text{antidiff.}}$$

$$f(x,y) = x^4 + 3x^3y^2 + h(y) \xrightarrow{\partial/\partial y}$$

$$\underline{6x^3y} + 6y^5 = f_y = 0 + \underline{6x^3y} + h'(y) \Longrightarrow$$

$$h'(y) = 6y^5 \Rightarrow h(y) = y^6 + c \Longrightarrow$$

$$f(x,y) = x^4 + 3x^3y^2 + y^6 + c$$ (so ω is exact)

Finally, for $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, consider the integral around $\mathcal{C}$:

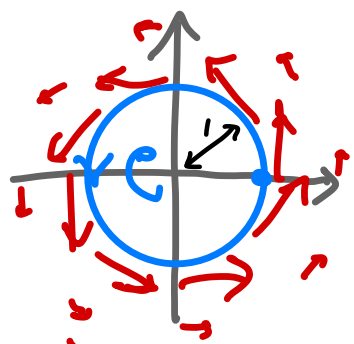
parametrizing by $\vec{r}(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$

gives $\oint_{\mathcal{C}} \omega = \int_0^{2\pi} \vec{r}^{\,*}\omega$

$$= \int_0^{2\pi} -\sin t\,(\cos'(t)\,dt) + \cos t\,(\sin'(t)\,dt)$$

$$= \int_0^{2\pi} \cancel{(\sin^2 t + \cos^2 t)}^{\,1}\,dt = 2\pi \neq 0,$$

so by the Proposition ω CANNOT be exact. We'll come back to this.



$$\vec{F} = \frac{1}{x^2+y^2}\begin{pmatrix}-y\\ x\end{pmatrix}$$

First write

$$d(P\,dx + Q\,dy) = P_y\,dy\wedge dx + Q_x\,dx\wedge dy$$
$$= (Q_x - P_y)\,dx\wedge dy.$$

(i) $Q_x - P_y = \frac{\partial}{\partial x}(6x^3y + 6y^5) - \frac{\partial}{\partial y}(4x^3 + 9x^2y^2)$

$$= 18x^2y - 18x^2y = 0$$

~~(ii)~~ $Q_x - P_y = \frac{\partial}{\partial x}(x^2 - y^2) - \frac{\partial}{\partial y}(xy^2)$

$$= 2x - 2xy \neq 0$$

(iii) $Q_x - P_y = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = 0$$

We already know exact $\Rightarrow$ closed, so not closed $\Rightarrow$ not exact. Hence (ii) is not exact.

For (i), we can suppose that ω is exact, $\omega = df = f_x\,dx + f_y\,dy$, and try to solve $f_x = P$, $f_y = Q$.

§2. Finding a potential function

How can we tell if a 1-form ω is exact (i.e. if $\vec{F}$ is conservative)?

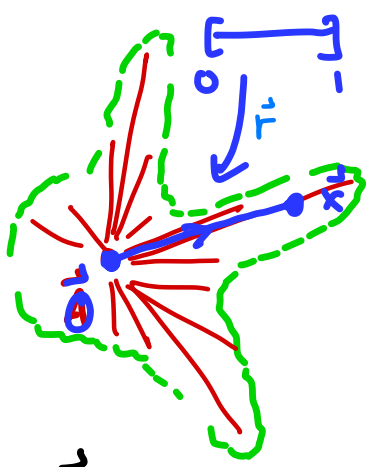
Well, if $\omega = df$, then

$$d\omega = d(df) = 0,$$

i.e. ω is closed. This is easy to check, so might it be the case that $\boxed{\text{closed} \Rightarrow \text{exact}}$? (*)

Suppose that $\omega \in \mathcal{A}^1(U)$, with U star-shaped:

that is, for some point $\vec{A} \in U$, every point of U is at the end of a line segment starting at $\vec{A}$.



I claim that in this case (*) holds:

$$0 \overset{\text{assume}}{=} d\omega = d\sum_{j=1}^{n} F_j\, dx_j = \sum_{i,j} \frac{\partial F_j}{\partial x_i} dx_i \wedge dx_j$$

$$= \sum_{i<j}\left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\right) dx_i \wedge dx_j$$

forces $\partial F_j/\partial x_i = \partial F_i/\partial x_j$ for all i,j.

For notational simplicity assume $\vec{A} = \vec{0}$, and put

$$f(\vec{x}) := \int_{\vec{0}}^{\vec{x}} \omega = \int_0^1 \vec{r}^{\,*}\omega$$

$$= \int_0^1 \sum_j F_j(t\vec{x})\, x_j\, dt$$

$\vec{r}(t) = t\vec{x}$, $(dx_i = x_i\, dt)$

$$= \sum_j x_j \int_0^1 F_j(t\vec{x})\, dt$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(\vec{x}) = \int_0^1 F_i(t\vec{x})\,dt + \sum_j x_j \int_0^1 t\,\frac{\partial F_j}{\partial x_i}(t\vec{x})\,dt$$

$$= \int_0^1 F_i(t\vec{x})\,dt + \int_0^1 t \underbrace{\sum_j x_j \frac{\partial F_j}{\partial x_i}(t\vec{x})}_{\frac{d}{dt}F_i(t\vec{x})\ \text{(chain rule)}}\,dt$$

$\int$ by parts

$$= t\,F_i(t\vec{x})\Big|_0^1$$

$$= F_i(\vec{x}).$$

So $df = \sum F_i(\vec{x})\,dx_i = \omega$.

The most general result is that $\omega \in A^1(U)$ closed ($d\omega = 0$) $\Rightarrow$ exact ($\omega = df$) as long as U is <u>simply connected</u>, i.e. has no holes:

good

I claim that in this case (*) holds:

$$0 \overset{\text{assume}}{=} d\omega = d\sum_{j=1}^{n} F_j\, dx_j = \sum_{i,j} \frac{\partial F_j}{\partial x_i} dx_i \wedge dx_j$$

$$= \sum_{i<j}\left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\right) dx_i \wedge dx_j$$

forces $\partial F_j/\partial x_i = \partial F_i/\partial x_j$ for all i,j.

For notational simplicity assume $\vec{A} = \vec{0}$, and put

$$f(\vec{x}) := \int_{\vec{0}}^{\vec{x}} \omega = \int_0^1 \vec{r}^{\,*}\omega$$

$$= \int_0^1 \sum_j F_j(t\vec{x})\, x_j\, dt$$

$\vec{r}(t) = t\vec{x}$

$(dx_i = x_i\, dt)$

$$= \sum_j x_j \int_0^1 F_j(t\vec{x})\, dt$$

$$\Rightarrow \frac{\partial f}{\partial x_i}(\vec{x}) = \int_0^1 F_i(t\vec{x})\,dt + \sum_j x_j \int_0^1 t\,\frac{\partial F_i}{\partial x_j}(t\vec{x})\,dt$$

$$= \int_0^1 F_i(t\vec{x})\,dt + \int_0^1 t \underbrace{\sum_j x_j \frac{\partial F_i}{\partial x_j}(t\vec{x})}_{\frac{d}{dt}F_i(t\vec{x})\ \text{(chain rule)}}\,dt$$

$\int$ by parts

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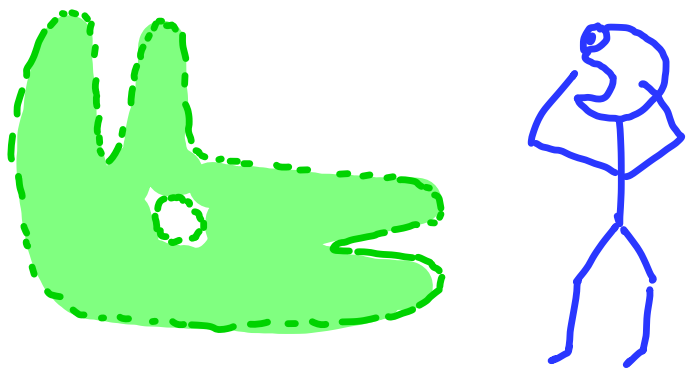
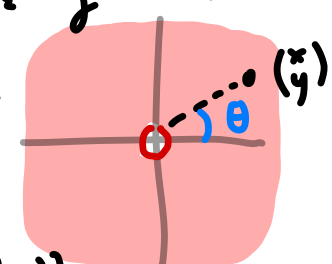
So $df = \sum F_i(\vec{x})\,dx_i = \omega$.

The most general result is that $\omega \in A^1(U)$ closed ($d\omega = 0$) $\Rightarrow$ exact ($\omega = df$) as long as U is <u>simply connected</u>, i.e. has no holes:

bad



This is precisely what goes wrong with $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$: it lives on $\mathbb{R}^2 \setminus \{\vec{0}\}$, which is not simply connected.



Locally it is $d(\underbrace{\arctan(y/x)}_{\text{i.e. "}\theta\text{"}})$, but θ does not yield a well-defined function globally on $\mathbb{R}^2 \setminus \{\vec{0}\}$ (going around the origin, it changes by 2π).

Ex/ Let $\mathcal{C}$ be some path in $\mathbb{R}^3$ from $\vec{A} = \begin{pmatrix} 0 \\ \pi \\ 2 \end{pmatrix}$ to $\vec{B} = \begin{pmatrix} 1 \\ \pi/2 \\ -3 \end{pmatrix}$, and

$$\omega = \underbrace{(e^x \cos y + yz)}_{P}dx + \underbrace{(xz - e^x \sin y)}_{Q}dy + \underbrace{(xy + 2z)}_{R}dz.$$

Find $\int_{\mathcal{C}} \omega$.

I'll just outline the solution:

- Check $d\omega = 0$.
- Since ω is defined on $\mathbb{R}^3$, which is certainly star-shaped, this implies $\omega = df$.
- find f by setting $f_x = P$, $f_y = Q$, $f_z = R$. We get $f = e^x \cos y + xyz + z^2$.
- apply the FTC:

$$\int_{\mathcal{C}} \omega = f(\vec{B}) - f(\vec{A}) = \left(9 - \frac{3\pi}{2}\right) - 3$$
$$= 6 - \frac{3\pi}{2}.$$

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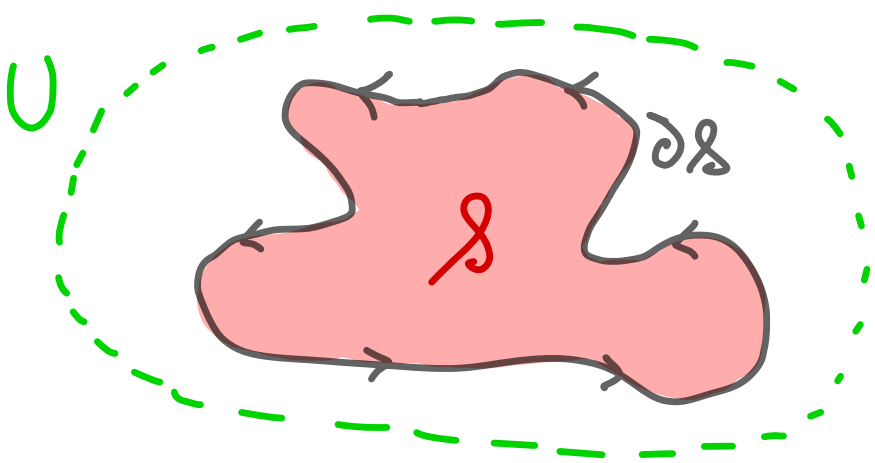
§3. Green's Theorem (in $\mathbb{R}^2$)

This is an analogue of the FTC

$$\int_C df = \text{"}\int_{\partial C} f\text{"} \quad (\text{i.e. } f(B)-f(A))$$

for integrals of 2-forms over a region.

Suppose $\omega = P(x,y)\,dx + Q(x,y)\,dy$ is any 1-form defined on an open set $U \subset \mathbb{R}^2$, and $S \subset U$ is a region with boundary curve ∂S piecewise C^1.



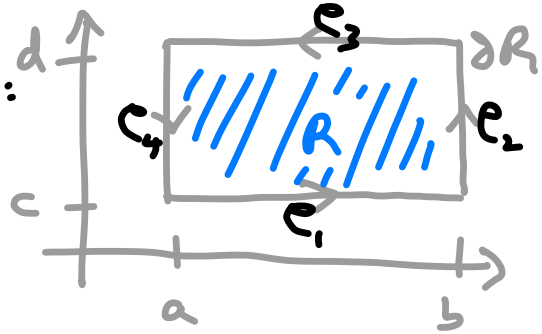
∂S is oriented so that S is "on the left" (counterclockwise orientation).

$$\boxed{\text{Theorem}: \int_S d\omega = \oint_{\partial S} \omega}$$

Notes:

- this is in the same spirit as the FTC, in that it uses boundary values of P & Q to compute the integral of $d\omega = (Q_x - P_y)\,dx\wedge dy$ on the interior.
- it implies the "closed $\Rightarrow$ exact" result for U simply connected, which we can define to mean that the region enclosed by a simple closed curve C is contained in U. Then ω closed ($d\omega = 0$) $\Rightarrow \int_S d\omega = 0$ $\Rightarrow \oint_C \omega = \oint_{\partial S} \omega = 0 \Rightarrow \omega$ exact.

Proof for rectangle:



$$\int_{\partial R} \omega =$$

$$\int_{C_1} \omega + \int_{C_2} \omega + \int_{C_3} \omega + \int_{C_4} \omega =$$

$$\int_a^b P\binom{x}{c}\,dx + \int_c^d Q\binom{b}{y}\,dy - \int_a^b P\binom{x}{d}\,dx - \int_c^d Q\binom{a}{y}\,dy$$

On the other side,

$$\int_R d\omega = \int_R (Q_x - P_y)\,dx \wedge dy$$

Fubini

$$= \int_c^d \int_a^b Q_x \,dx\,dy - \int_a^b \int_c^d P_y \,dy\,dx$$

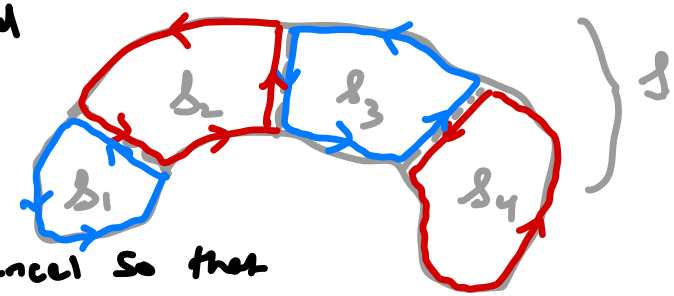
$$= \int_c^d Q\binom{b}{y}\,dy - \int_c^d Q\binom{a}{y}\,dy$$

$$- \left(\int_a^b P\binom{x}{d}\,dx - \int_a^b P\binom{x}{c}\,dx \right),$$

which equals the expression for $\int_{\partial R} \omega$.

$$\boxed{\text{Theorem}: \int_S d\omega = \oint_{\partial S} \omega}$$

Proof for general region: Break S up into little subregions parametrized by rectangles and observe that internal boundaries cancel so that $\sum \partial S_i = \partial S$. So it suffices to prove for S the image of a 1-1 C^1 map $\vec{G}: R \to S$. But



$$\underset{\vec{G}(\partial R)}{\overset{\shortparallel}{\int_{\partial S}}} \omega = \int_{\partial R} \vec{G}^* \omega \underset{\text{Green for rectangle}}{=} \int_R d(\vec{G}^* \omega)$$

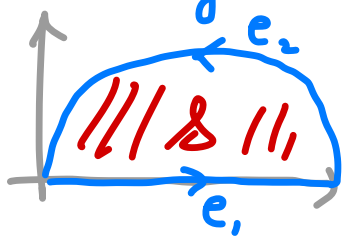
$$= \int_R \vec{G}^*(d\omega) = \underset{\vec{G}(R)}{\overset{\shortparallel}{\int_S}} d\omega.$$

$\square$

Ex/ Find the area under one arch of the cycloid parametrized by

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t - \sin(t) \\ 1 - \cos(t) \end{pmatrix}.$$

$(t \in [0, 2\pi])$



If $\omega = x\,dy$, then $d\omega = dx \wedge dy$.

So $\text{Area}(\mathcal{S}) = \int_{\mathcal{S}} dx \wedge dy \underset{\text{Green}}{=} \oint_{\partial \mathcal{S}} x\,dy$

$$= \cancel{\int_{C_1} x\,dy} + \int_{C_2} x\,dy \underset{\text{use } \vec{r}(t)}{=} \underset{\text{orientation}}{-} \int_0^{2\pi} \vec{r}^{\,*}(x\,dy)$$

0 b/c dy pulls back to 0

$$= -\int_0^{2\pi} x(t)\, y'(t)\, dt = \int_0^{2\pi} (\sin(t) - t)\sin(t)\, dt$$

$$= \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2} - t \sin t \right) dt$$

$$= \left[\frac{t}{2} - \frac{1}{4}\sin 2t - \sin t + t\cos t \right]_0^{2\pi}$$

$$= 3\pi.$$

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$$\boxed{\text{Theorem}: \int_{\mathcal{S}} d\omega = \oint_{\partial \mathcal{S}} \omega}$$

Ex/ Find the line integral

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} \quad \omega$$

$\partial \mathcal{S} = C - C'$

and we know that

$$\oint_{C'} \omega = 2\pi.$$

We also know $d\omega = 0$,

so

$$0 = \int_{\mathcal{S}} d\omega = \int_{C - C'} \omega = \int_C \omega - \int_{C'} \omega$$

$$= \int_C \omega - 2\pi$$

$$\Rightarrow \oint_C \omega = 2\pi.$$

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