# Surface Integrals

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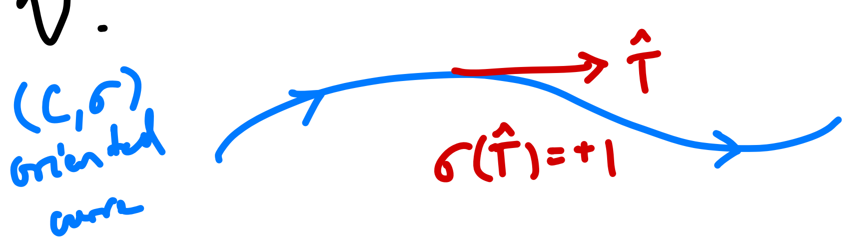
Say we want to integrate a $k$-form $\omega$ over a $k$-manifold $M \subset \mathbb{R}^n$.

You need a parametrization

$$\vec{g} : \underset{\mathbb{R}^k}{\underset{\cap}{\Omega}} \longrightarrow M,$$

and then $\int_M \omega := \int_\Omega \vec{g}^{\,*} \omega$.

If $\vec{h} : \mathcal{V} \longrightarrow M$ is another, then by Lecture 22 $\int_{\mathcal{V}} \vec{h}^{\,*} \omega$ gives the same answer... up to sign: it is $\pm \int_\Omega \vec{g}^{\,*}\omega$, depending on whether $\det D(\vec{g}^{-1} \circ h) > 0$ or $< 0$ on $\mathcal{V}$.




We saw this with curves, where changing the direction of the parametrization changes the sign of $\int_C \omega$.

So to make $\int_C \omega$ "well-defined", we needed to choose an "orientation" of $C$, and require $\vec{g}$ to respect it.

One way to do this is to fix a 1-form $\sigma$ on (a neighborhood of) the curve, which on every unit tangent vector $\hat{T}$ to $C$ has $\sigma(\hat{T}) = \pm 1$.

The "+" direction is the direction of the oriented curve $(C, \sigma)$, and a parametrization $\vec{g}$ must have $g^* \sigma = f(t)\,dt$ with $f(t) > 0$.

## §1. Oriented surfaces

We are fussing about this because we want the integral of a 2-form over a smooth surface to be a well-defined number.

The setup is always this:

$$\begin{array}{ccccc} \mathcal{S} & \subset & V & \subset & \mathbb{R}^n \\ \uparrow \vec{r} & & \uparrow \vec{r} & & \\ \text{region } \Omega & \subset & U & \subset & \mathbb{R}^2 \end{array}$$

⟵ $D\vec{r}$ everywhere of rank 2

Given $\omega \in A^2(V)$,

$$\int_{\mathcal{S}} \omega := \int_{\Omega} \vec{r}^{\,*}\omega .$$

The standard computation for the pullback is this (when $n=3$):

$$\vec{r}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix} x\binom{u}{v} \\ y\binom{u}{v} \\ z\binom{u}{v} \end{pmatrix}$$

with partials $\vec{r}_u = \begin{pmatrix}x_u\\y_u\\z_u\end{pmatrix}$, $\vec{r}_v = \begin{pmatrix}x_v\\y_v\\z_v\end{pmatrix}$

$$r^* dx\wedge dy = (x_u\,du + x_v\,dv)\wedge(y_u\,du + y_v\,dv)$$

$$= (x_u y_v - x_v y_u)\,du\wedge dv$$

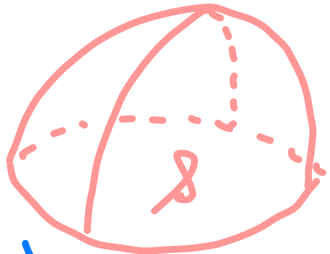
$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv$$

$$= \left(dx\wedge dy\,(\vec{r}_u, \vec{r}_v)\right) du\wedge dv$$

So for example if $\omega = z\,dx\wedge dy$,

$$\vec{r}^{\,*}\omega = z\begin{pmatrix}u\\v\end{pmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv .$$

## $\boxed{\text{Ex 1a}}$ $\omega = z\,dx\wedge dy$




$$\vec{r} : \underbrace{[0,1]\times[0,2\pi]}_{\Omega} \to \mathbb{R}^3$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u\cos v \\ u\sin v \\ \sqrt{1-u^2} \end{pmatrix}$$

$$\vec{r}^*\omega = \sqrt{1-u^2}\begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} du\wedge dv$$

$$= u\sqrt{1-u^2}\, du\wedge dv$$

$$\int_S \omega = \int_\Omega \vec{r}^*\omega = \int_0^{2\pi}\int_0^1 u\sqrt{1-u^2}\,du\,dv$$

$$= \left(\int_0^{2\pi} dv\right)\left(\left[-\frac{1}{3}(1-u^2)^{3/2}\right]_0^1\right) = \frac{2\pi}{3}.$$

## $\boxed{\text{Ex 1b}}$ same $\omega$ & $S$

$$\vec{r} : \left[0,\frac{\pi}{2}\right]\times[0,2\pi] \to \mathbb{R}^3$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} \sin u\cos v \\ \sin u\sin v \\ \cos u \end{pmatrix}$$

$$r^*\omega = \cos u\begin{vmatrix} \cos u\cos v & -\sin u\sin v \\ \cos u\sin v & \sin u\cos v \end{vmatrix} du\wedge dv$$

$$= \cos u\left(\cos u\sin u\,\{\cancel{\cos^2 v+\sin^2 v}\}^{1}\right) du\wedge dv$$

$$= \cos^2 u\,\sin u\;du\wedge dv$$

$$\int_S \omega = \int_\Omega \vec{r}^*\omega = \int_0^{2\pi}\int_0^{\pi/2}\cos^2 u\sin u\,du\,dv$$

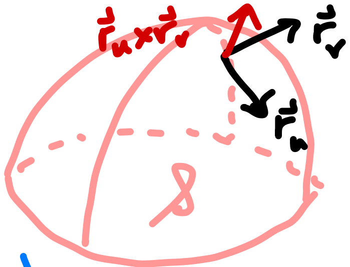
$$= \left(\int_0^{2\pi} dv\right)\left(\left[-\frac{1}{3}\cos^3 u\right]_0^{\pi/2}\right)$$

MAGIC!

$$= \frac{2\pi}{3}.$$

$$\boxed{\vec{r}^*\omega = z\begin{pmatrix} u \\ v \end{pmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv.}$$

## Ex 1a  $\omega = z\,dx\wedge dy$




$\vec r : \underbrace{[0,1]\times[0,2\pi]}_{\Omega} \to \mathbb{R}^3$

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## Ex 1b  same $\omega$ & $S$

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$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} \sin u\cos v \\ \sin u\sin v \\ \cos u \end{pmatrix}$$

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$$= \cos u\left(\cos u\sin u \cancelto{1}{\{\cos^2 v+\sin^2 v\}}\right) du\wedge dv$$

$$= \cos^2 u\,\sin u\;du\wedge dv$$

$$\int_S \omega = \int_\Omega \vec r^{\,*}\omega = \int_0^{2\pi}\int_0^{\pi/2} \cos^2 u\sin u\,du\,dv$$

$$= \left(\int_0^{2\pi} dv\right)\left(\left[-\frac{1}{3}\cos^3 u\right]_0^{\pi/2}\right)$$

$$= \frac{2\pi}{3}.$$

MAGIC!

## Ex 1c  same $\omega$ & $S$


$\vec r : [0,1]\times[0,2\pi] \to \mathbb{R}^3$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u\sin v \\ u\cos v \\ \sqrt{1-u^2} \end{pmatrix}$$

changes sign of $\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$r^*\omega = -u\sqrt{1-u^2}\,du\wedge dv$$

$$\Longrightarrow \int_S \omega = -\frac{2\pi}{3}.$$ BAD

The problem is that we are not integrating over the same oriented surface. In particular, the normal vector $\vec r_u \times \vec r_v$ has changed direction.

What can we do to fix the sign problem? Geometrically, you need to fix a continuous choice of unit normal all over $S$ * and insert the parametrization's $\vec{r}_u \times \vec{r}_v$ point in this direction. You could also choose a 2-form $\sigma$ on $S$ which, applied to any orthonormal basis $\vec{u}, \vec{v}$ of each tangent plane, yields $\sigma(\vec{u}, \vec{v}) = \pm 1$. This is called an area 2-form, and there are only two choices.

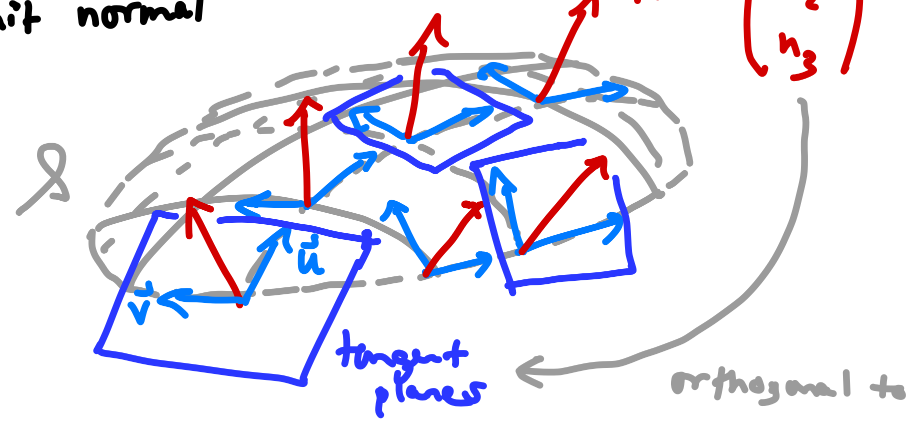
* assuming that's is possible, i.e. $S$ isn't something like the Möbius band

What can we do to fix the sign problem? Geometrically, you need to fix a continuous choice of unit normal all over $\mathcal{S}$ * and insist the parametrization's $\vec{r}_u \times \vec{r}_v$ point in this direction. You could also choose a 2-form $\sigma$ on $\mathcal{S}$ which, applied to any orthonormal basis $\vec{u}, \vec{v}$ of each tangent plane, yields $\sigma(\vec{u}, \vec{v}) = \pm 1$. This is called an area 2-form, and there are only two choices. The oriented surface $(\mathcal{S}, \sigma)$ is then parametrized by $\vec{r} : \Omega \to \mathcal{S}$ if $\vec{r}^* \sigma = h(u,v)\, du \wedge dv$ with $h > 0$ on $\Omega$.

## *2. Surface area

But if there is an "area form" $\sigma$, shouldn't integrating it be the right definition of area of $\mathcal{S}$? Continuing to assume $\mathcal{S}$ lives in $\mathbb{R}^3$, here is how we can write $\sigma$ down: start with a continuously varying unit normal

$$\hat{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$



$\{\vec u, \vec v\}$ of the tangent plane $T_p S$:

$$\sigma(\vec u, \vec v) = n_1 \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} + n_2 \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix} + n_3 \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

$$= \hat n \cdot (\vec u \times \vec v) = \hat n \cdot \pm \|\vec u \times \vec v\| \hat n$$

$$= \pm \|\vec u \times \vec v\| = \text{oriented area of } P_{\vec u, \vec v}$$

If $\vec u$ & $\vec v$ are orthonormal, this is certainly $\pm 1$! So $\sigma$ is the area form corresponding to the choice of direction of $\hat n$.

Now let $\vec r : \Omega \to S$ be a parametrization of $(S, \sigma)$. Define

$$\boxed{\text{Area}(S) := \int_S \sigma = \int_\Omega \vec r^* \sigma.}$$

How can we compute this?

## §2. Surface area

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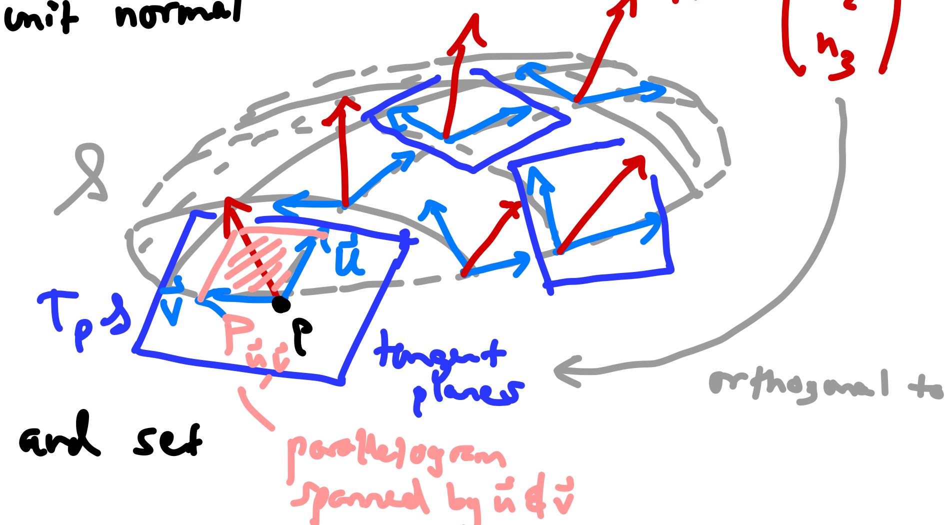


and set

$$\sigma := n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy.$$

Applying this at $p$ to a basis

$\{\vec{u},\vec{v}\}$ of the tangent plane $T_p\mathcal{S}$ :

$$\sigma(\vec{u},\vec{v}) = n_1\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3\end{vmatrix} + n_2\begin{vmatrix} u_3 & v_3 \\ u_1 & v_1\end{vmatrix} + n_3\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2\end{vmatrix}$$

$$= \hat{n}\cdot(\vec{u}\times\vec{v}) = \hat{n}\cdot \pm\|\vec{u}\times\vec{v}\|\,\hat{n}$$

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Now let $\vec{r}:\Omega\to\mathcal{S}$ be a parametrization of $(\mathcal{S},\sigma)$. Define

$$\boxed{\text{Area}(\mathcal{S}) := \int_{\mathcal{S}}\sigma = \int_{\Omega}\vec{r}^{\,*}\sigma.}$$

How can we compute this?

One way would be to make $\sigma$ explicit. For example, if $\mathcal{S}$ = the unit sphere, $\vec{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is a unit (outward) normal at every point on $\mathcal{S}$. So we get

$$\sigma = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy.$$

But it is rarely this nice and pulling even this $\sigma$ back can be a long computation.

Better: observe that

$$\vec{r}^{\,*}\sigma = r^*(n_1\,dy\wedge dz + n_2\,dz\wedge dx + n_3\,dx\wedge dy)$$

$$= \left\{ (n_1\circ\vec{r})\begin{vmatrix} y_u & y_v \\ z_u & z_v\end{vmatrix} + (n_2\circ\vec{r})\begin{vmatrix} z_u & z_v \\ x_u & x_v\end{vmatrix} + (n_3\circ\vec{r})\begin{vmatrix} x_u & x_v \\ y_u & y_v\end{vmatrix}\right\} du\wedge dv$$

(by computation $d_m$ §1)

$$= \overbrace{\hat{n}\cdot(\vec{r}_u\times\vec{r}_v)}^{>0 \text{ by assumption}}\,du\wedge dv = \|\vec{r}_u\times\vec{r}_v\|\,du\wedge dv$$

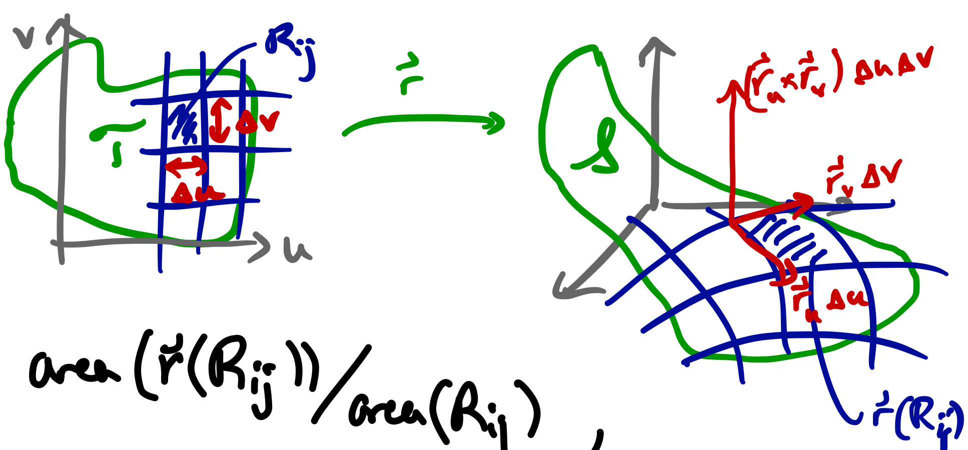
So the formula

$$\boxed{\text{Area}(\mathcal{S}) := \int_{\mathcal{S}} \sigma = \int_\Omega \vec{r}^{\,*}\sigma}$$

becomes

$$\int_\Omega \|\vec r_u \times \vec r_v\| \, du \, dv .$$

The intuition here is that $\|\vec r_u \times \vec r_v\|$ should be viewed as the ratio




$$\text{area}(\vec r(R_{ij}))/\text{area}(R_{ij}),$$

just as in the change-of-variable formula.

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Better: observe that

$$\vec r^{\,*}\sigma = r^*(n_1 dy\wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy)$$

$$= \left\{ (n_1 \circ \vec r) \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} + (n_2 \circ \vec r) \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} + (n_3 \circ \vec r) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\} du \wedge dv$$

(by computation in §1)

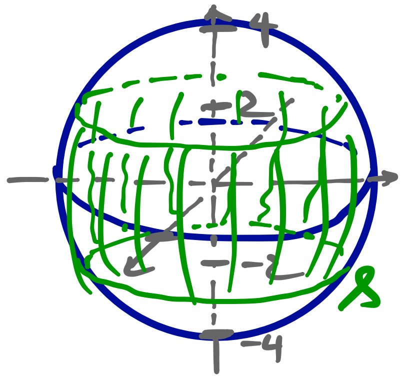
$$= \overbrace{\hat n \cdot (\vec r_u \times \vec r_v)}^{>0 \text{ by assumption}} \, du \wedge dv = \|\vec r_u \times \vec r_v\| \, du \wedge dv$$

hence $\|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix} \right\| = \sqrt{1+f_u^2+f_v^2}$

and

$$\text{Area}(S) = \int_D \sqrt{1+f_u^2+f_v^2}\, du\, dv. \quad //$$

**PROBLEM:** Compute the surface area of the part of the sphere of radius 4 between $z=-2$ & $z=2$.




We must limit $4\cos u (=z)$ to between $+2$ & $-2$, i.e. $\cos u \in [-\frac{1}{2}, \frac{1}{2}] \Rightarrow u \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. Otherwise $\vec{r}$ same as in Ex. 2. So

$$\text{Area}(S) = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} 4^2 \sin u\, du\, dv$$

$$= 32\pi \int_{\pi/3}^{2\pi/3} \sin u\, du = 32\pi \underbrace{\left[-\cos u\right]_{\pi/3}^{2\pi/3}}_{\frac{1}{2}-(-\frac{1}{2})}$$

$$= 32\pi.$$

$\boxed{\text{Ex 2}}$ We compute the surface area of the upper half of a sphere of radius $R$. & 

Take $\vec{r}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} R\sin u \cos v \\ R \sin u \sin v \\ R\cos u \end{pmatrix}$, $\left[\begin{matrix} 0 \le u \le \pi/2 \\ 0 \le v \le 2\pi \end{matrix}\right]$ and

compute $\|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{pmatrix} R^2 \sin^2 u \cos v \\ R^2 \sin^2 u \sin v \\ R^2 \cos u \sin u \end{pmatrix} \right\| = R^2 \sin u$ $(>0)$

Then $\text{Area}(S) = \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin u\, du\, dv$

$$= 2\pi R^2 \int_0^{\pi/2} \sin u\, du = 2\pi R^2 \quad //$$

$\boxed{\text{Ex 3}}$ What about the surface area of a graph of $z = f(x,y)$ over $D \subset \mathbb{R}^2$:


Use the parametrization

$\vec{r}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$, with

$\vec{r}_u = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}$, $\vec{r}_v = \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}$

## §3. Flux through a surface

Given $\vec{r} : \Omega \to \mathcal{S}$ a parametric (oriented) surface, we have seen that

$$\text{Area}(\mathcal{S}) = \int_\Omega \|\vec{r}_u \times \vec{r}_v\| \, dA .$$

Given a function $f : \mathcal{S} \to \mathbb{R}$, we can also define its integral over $\mathcal{S}$ "with respect to surface area" by

$$\int_{\mathcal{S}} f \, dS := \int_\Omega (f \circ \vec{r}) \|\vec{r}_u \times \vec{r}_v\| \, dA .$$

If $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ is a vector field on (a set containing) $\mathcal{S}$, we are interested in the integral of $\vec{F} \cdot \hat{n}$ over $\mathcal{S}$. Why?

Suppose we have on $\mathbb{R}^3$

- $\vec{V}$ = fluid velocity field   m/sec.
- $\delta$ = fluid density function   $\text{kg}/\text{m}^3$
- $\vec{F} = \delta \vec{V}$ = flux density   $\dfrac{\text{kg}/\text{sec.}}{\text{m}^2}$

Then $\vec{F} \cdot \hat{n}$ is the flux density in the normal direction, and the flux through $\mathcal{S}$

- $\int_{\mathcal{S}} \vec{F} \cdot \hat{n} \, dS$ is measured in kg/sec.

There are two ways of thinking about this. First,

$$\int_{\mathcal{S}} \vec{F} \cdot \hat{n} \, dS = \int_\Omega \vec{F}\left(\vec{r}\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}}_{\hat{n}} \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, dA$$

$$= \int_\Omega \underbrace{\vec{F}\left(r\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot (\vec{r}_u \times \vec{r}_v)}_{\text{scalar triple product}} \, du \, dv$$

$$\begin{vmatrix} \uparrow & \uparrow & \uparrow \\ \vec{F} & \vec{r}_u & \vec{r}_v \\ \downarrow & \downarrow & \downarrow \end{vmatrix}$$

This is the most useful formula for computation (usually). But now notice that the 2-form

$$\eta_{\vec{F}} = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$$

corresponding to $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$ has

$$\vec{r}^{\,*}\eta_{\vec{F}} = \left\{ (F_1 \circ \vec{r}) \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} + (F_2 \circ \vec{r}) \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} + (F_3 \circ \vec{r}) \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \right\} du \wedge dv$$

$$= \underbrace{\begin{vmatrix} F_1 & x_u & x_v \\ F_2 & y_u & y_v \\ F_3 & z_u & z_v \end{vmatrix}}_{\text{Same scalar triple-product!}} du \wedge dv .$$

So

$$\int_S \vec{F} \cdot \hat{n}\, dS = \int_S \eta_{\vec{F}} ,$$

and this is the nicest theoretical definition of flux

$\boxed{\text{Ex 4}}$ Find the flux of $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ through the unit sphere $S$.

We compute the scalar triple product (with $\vec{r}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{pmatrix}$ as usual)

$$\underbrace{\vec{F}\left(\vec{r}\begin{pmatrix} u \\ v \end{pmatrix}\right)}_{= r\binom{u}{v}} \cdot \left(\vec{r}_u \times \vec{r}_v\right) = \begin{vmatrix} \sin u \cos v & \cos u \cos v & -\sin u \sin v \\ \sin u \sin v & \cos u \sin v & \sin u \cos v \\ \cos u & -\sin u & 0 \end{vmatrix}$$

$$= \cos u (\cos u \sin u) + \sin u (\sin^2 u)$$

$$= \sin u (\cos^2 u + \sin^2 u) = \sin u .$$

So

$$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} \sin u \, du\, dv$$

$$= 2\pi \int_0^{\pi} \sin u \, du$$

$$= 4\pi .$$

//