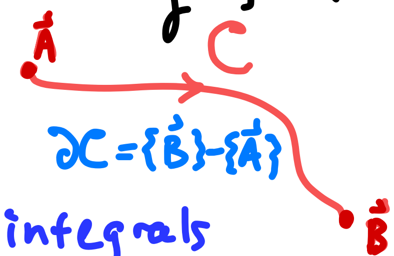
Stokes's Theorem 25

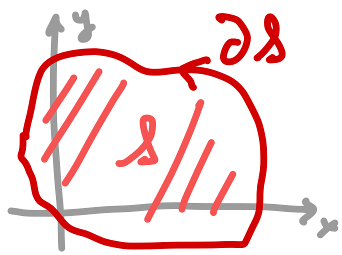
So far we have encountered two main results relating integrals on the interior and the boundary of a manifold.



① FTC for line integrals

$$\int_C df = \text{``}\int_{\partial C} f\text{''} := f(\vec{B}) - f(\vec{A})$$

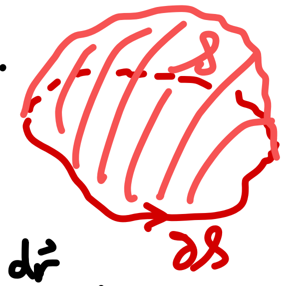
② Green's theorem



$$\int_S d\omega = \oint_{\partial S} \omega \qquad (\omega \text{ 1-form})$$

In multivariable calculus courses like Math 233,

②' Stokes's theorem is the name given to the generalization of ② to S = a surface in $\mathbb{R}^3$.



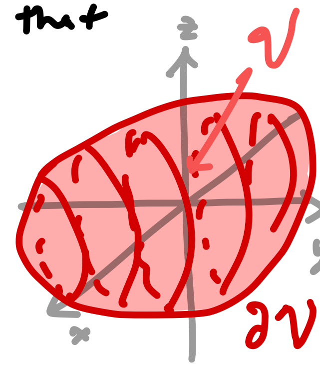
It is usually written

$$\int_S (\text{curl } \vec{F}) \cdot \hat{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r},$$

just as ① is written (in that course) as

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{B}) - f(\vec{A}).$$

③ Gauss's theorem says that for a solid V we have



$$\int_V d\omega = \int_{\partial V} \omega$$

for ω any 2-form. You may have seen it in the form

$$\int_V \text{div } \vec{F}\, dV = \int_{\partial V} \vec{F} \cdot \hat{n}\, dS.$$

The moral is that, while in their "vector" form these look like very different results, they look like "the same" result when expressed in terms of differential forms. Namely:

(GENERALIZED) STOKES'S THEOREM

If $M \subset \mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then

$$\int_M d\omega = \int_{\partial M} \omega .$$

Once you know what forms are, this is easier to prove than ② & ③ directly, but it immediately implies them.

§1. Integration on manifolds revisited

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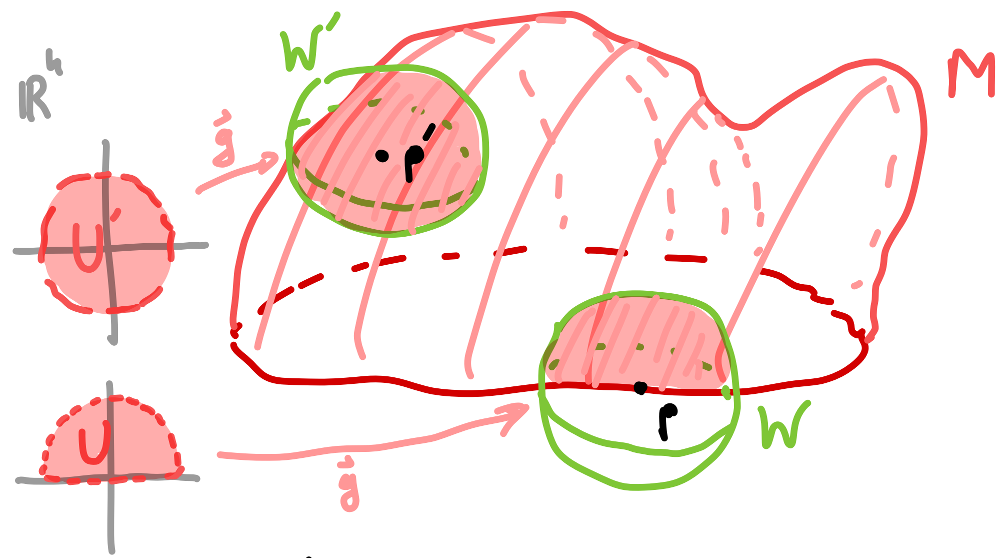
The main observation we need is that it is possible to break an integral

$$\int_M \eta$$

k-manifold in $\mathbb{R}^n$ → ← k-form

up into integrals over subsets parametrized by balls $B_i \subset \mathbb{R}^k$. We are going to assume that about every $p \in M$, there exists a neighborhood $W \subset \mathbb{R}^n$

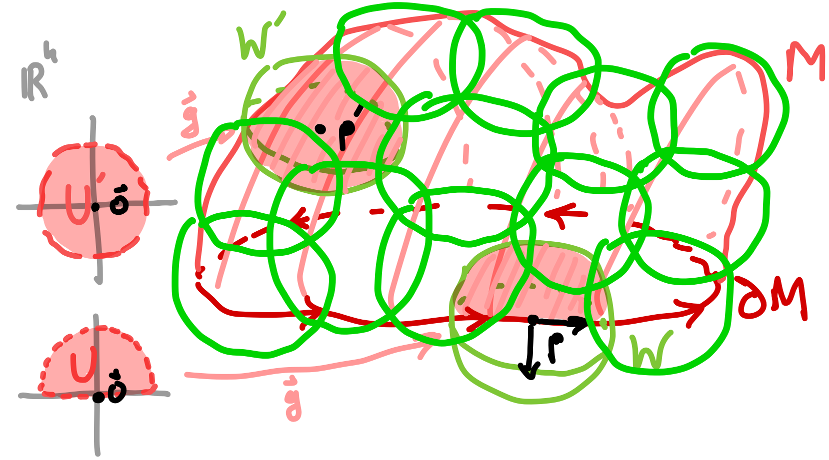


and a C^∞ parametrization $\vec{\varphi} : U \to W \cap M$

"coordinate charts"

Where $U \subset \mathbb{R}^k$ is either an open ball centered at the origin or its intersection with $\underset{u_1,\dots,u_{k-1}}{\mathbb{R}^{k-1}} \times \underset{u_k}{\mathbb{R}_{\geq 0}} =: \mathbb{R}^k_+$.

We say $\vec{p} \in \partial M$ (boundary points) if we are in the 2nd case, with $\vec{p}$ in the image of $\partial \mathbb{R}^k_+ := \mathbb{R}^{k-1} \times \{0\}$ via $\vec{g}$. We assume that these $\vec{g}$'s are compatible with a choice of orientation of M, so that the image by $D\vec{g}$ of $\hat{e}_1, \dots, \hat{e}_k$ is a "positively oriented" basis for T_pM. Give ∂M the orientation for which $D\vec{g}$ of $\hat{e}_1, \dots, \hat{e}_{k-1}$ is "$(-1)^k$-oriented" (i.e. positive if k=even & negative if k=odd); that is, the volume form pulls back to $(-1)^k du_1 \wedge \dots \wedge du_{k-1}$.

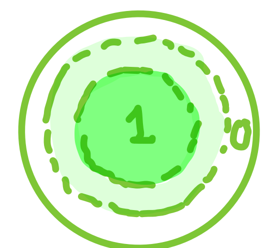


Since M is compact, it is covered with finitely many of these coordinate charts

$$\vec{g}_i : \underset{(\text{or } B_i)}{U_i} \to V_i \subset M.$$

Now argue that there are C^∞ functions

$$\varphi_i : V_i \to [0,1],$$



going to 0 at ∂V_i, which satisfy $\sum_i \varphi_i \equiv 1$ on M. These "overlapping bump functions" are constructed in Shifrin using $e^{-1/x}$, and are referred to as a partition of unity.

We can use the $\{\varphi_i\}$ to break integrals on M up into pieces:

$$\int_M \eta = \int_M \Big(\overbrace{\sum_i \varphi_i}^{=1}\Big) \eta$$

$$= \sum_i \int_{V_i} \varphi_i \eta \qquad \text{call this } \eta_i$$

$$= \sum_i \int_{U_i} g_i^* \eta_i \,.$$

This reduces the proof of Stokes's theorem to a local computation since we can write

$$\int_M d\omega = \sum_i \int_{V_i} d\omega_i$$

"local Stokes" $\rightarrow$ $$= \sum_i \int_{\partial V_i} \omega_i$$

$$= \int_{\partial M} \omega \,.$$

§2. The proof of Stokes

If $M \subset \mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then

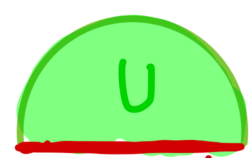
$$\int_M d\omega = \int_{\partial M} \omega \,.$$

By virtue of the argument in §1, we may assume M is covered by a single coordinate chart $g: U \to M$. There are two cases:

(I)

$(\partial M = \emptyset)$

(II)



$U \cap \partial \mathbb{R}^k_+$

(maps to ∂M)

§2. The proof of Stokes

> If $M \subset \mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then
>
> $$(*) \qquad \int_M d\omega = \int_{\partial M} \omega .$$

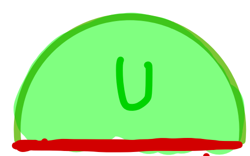
By virtue of the argument in §1, we may assume M is covered by a single coordinate chart $g: U \to M$. There are two cases, in which $(*)$ pulls back to

(I)

$(\partial M = \emptyset)$

$$\underbrace{\int_U g^* d\omega}_{\text{i.e. } \int_U d(g^*\omega)} = 0$$

(II)



$U \cap \partial \mathbb{R}^k_+$

(maps to ∂M)

$$\underbrace{\int_U g^* d\omega}_{\int_U d(g^*\omega)} = \int_{U \cap \partial \mathbb{R}^k_+} g^*\omega$$

The general form of a $(k-1)$-form on $U \subset \mathbb{R}^k$ is

$$g^*\omega = \sum_{i=1}^{k} f_i(\vec{u})\, du_1\wedge\cdots\wedge \overset{\text{"omit"}}{\widehat{du_i}}\wedge\cdots\wedge du_k$$

$$\Rightarrow d(g^*\omega) = \sum_{i=1}^{k} (-1)^{i-1}\frac{\partial f_i}{\partial u_i}\, du_1\wedge\cdots\wedge du_k .$$

(We are also assuming ω is 0 outside the dotted region in each picture.)

All we need to do at this point is check $(\dagger)$ and $(\dagger\dagger)$, which we do by enclosing U in a rectangle

$$R = [a_1,b_1]\times\cdots\times[\underset{=0 \text{ in case (II)}}{a_k}, b_k].$$

First compute $\underline{\text{RHS}(\dagger\dagger)}$: since $u_k = 0$,

use orientation:

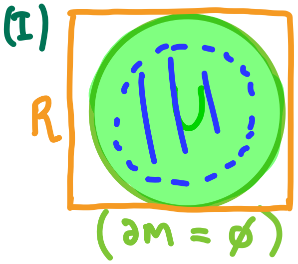
$$\int_{U\cap\partial\mathbb{R}^k_+} g^*\omega = \int_{U\cap\partial\mathbb{R}^k_+} f_k\begin{pmatrix} u_1\\ \vdots \\ u_{k-1}\\ 0\end{pmatrix} \underbrace{du_1\wedge\cdots\wedge du_{k-1}}_{(-1)^k\, dV_{k-1}}$$

$$= (-1)^k \int_{U\cap\partial\mathbb{R}^k_+} f_k\begin{pmatrix} u_1\\ \vdots \\ u_{k-1}\\ 0\end{pmatrix} dV_{k-1}$$

Fubini

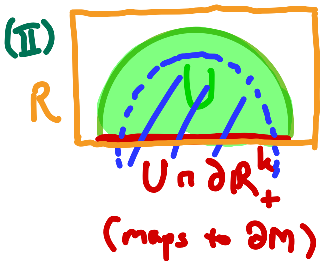
$$= (-1)^k \int_{a_{k-1}}^{b_{k-1}}\cdots\int_{a_1}^{b_1} f_k\begin{pmatrix} u_1\\ \vdots \\ u_{k-1}\\ 0\end{pmatrix} du_1\cdots du_{k-1} .$$

(I)



$$\int_U d(g^*\omega) = 0 \qquad (\dagger)$$

(II)



$$\int_U d(g^*\omega) = \underline{\int_{U\cap\partial\mathbb{R}^k_+} g^*\omega} \qquad (\dagger\dagger)$$

Next compute the LHS's: for both,

$$\int_U d(g^*\omega) = \int_U \left(\sum_{i=1}^{k}(-1)^{i-1}\frac{\partial f_i}{\partial u_i}\right) du_1\cdots du_k$$

Fubini

$$= \sum_{i=1}^{k}(-1)^{i-1}\int_{a_k}^{b_k}\cdots\int_{a_1}^{b_1}\left(\int_{a_i}^{b_i}\frac{\partial f_i}{\partial u_i}\,du_i\right) du_1\cdots\widehat{du_i}\cdots du_k$$

$$= \sum_{i=1}^{k}(-1)^{i-1}\int_{a_k}^{b_k}\cdots\int_{a_1}^{b_1}\left(f_i\begin{pmatrix} u_1\\ \vdots\\ b_i\\ \vdots\\ u_k\end{pmatrix} - f_i\begin{pmatrix} u_1\\ \vdots\\ a_i\\ \vdots\\ u_k\end{pmatrix}\right) du_1\cdots\widehat{du_i}\cdots du_k$$

✓ $= 0$ in case (I) $\Leftarrow$ but ω, hence the f_i, are zero on ∂R except in case (II) with $u_k = 0$.

in case (II) $\Leftarrow$

✓ $$= (-1)^{k-\cancel{1}}\int_{a_{k-1}}^{b_{k-1}}\cdots\int_{a_1}^{b_1}\left(\cancel{-}f_k\begin{pmatrix} u_1\\ \vdots\\ u_{k-1}\\ 0\end{pmatrix}\right) du_1\cdots du_{k-1}$$

which completes the proof of Stokes. $\square$

§3. Applications and examples

An immediate consequence is that if M has no boundary (like a closed curve, or a sphere), then the integral of any exact form $\int_M d\omega$ is zero. This is the higher-dimensional generalization of $\oint_C \vec{\nabla} f \cdot d\vec{r} = 0$ for the integral of a conservative force field around a loop.

In the event that ω is defined and closed ($d\omega = 0$) on all of M, we have $\int_{\partial M} \omega = 0$.

Ex 1/ $\omega = \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$

is defined on $\mathbb{R}^3 \setminus \{\vec{0}\}$, and $d\omega = 0$.

Since $r^2\omega$ is the area form on a sphere S_r of radius r about $\vec{0}$,

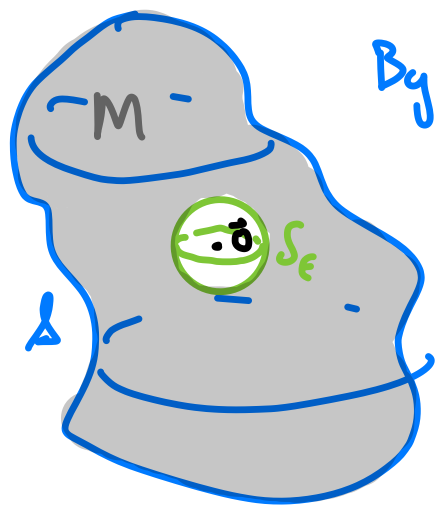
$$\int_{S_r} \omega = \frac{1}{r^2}\int_{S_r} r^2\omega = \frac{1}{r^2}\text{area}(S_r) = 4\pi.$$

(GENERALIZED) STOKES'S THEOREM

If $M \subset \mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

If $\mathcal{S} \subset \mathbb{R}^3$ is a closed surface (i.e. $\partial\mathcal{S} = \emptyset$) enclosing $\vec{0}$, then for $r = \epsilon$ sufficiently small it encloses S_ϵ too, and there is a solid M with $\partial M = \mathcal{S} - S_\epsilon$.



By Stokes's theorem,

$$0 = \int_{\partial M} \omega = \int_{\mathcal{S}} \omega - \int_{S_\epsilon} \omega = \int_{\mathcal{S}} \omega - 4\pi$$

$$\Longrightarrow \int_{\mathcal{S}} \omega = 4\pi. \quad //$$

Ex 2/ Find the flux of $\vec{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ through the unit sphere S.

Recall that this is

$$\int_S \underbrace{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}_{\omega},$$

and we did it by direct computation in the last lecture (with answer 4π).

Regarding $S = \partial M$ for M the solid unit ball $\{x^2+y^2+z^2 \le 1\}$, we can do this much more quickly now:

$$= \int_{\partial M} \omega = \int_M d\omega = \int_M 3\,dx\wedge dy\wedge dz$$

$$= 3\int_M dV = 3\cdot\frac{4}{3}\pi(1)^3$$

$$= 4\pi. \qquad //$$

(Generalized) Stokes's Theorem

If $M \subset \mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then

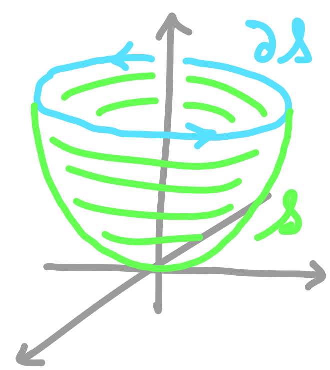
$$\int_M d\omega = \int_{\partial M} \omega.$$

Ex 3/ Check Stokes for

$\omega = y\,dx - x\,dy + yz\,dz,$

$S = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, z \le 1,\ z = x^2+y^2\right\},$

$\partial S = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, z = 1,\ x^2+y^2 = 1\right\}.$



$$\int_{\partial S} \omega = \oint_{\partial S} y\,dx - x\,dy + yz\,\underbrace{dz}_{\text{pulls back to } 0}$$

$$\Big\updownarrow g^*$$

$$= \int_0^{2\pi} \sin t\cdot(-\sin t\,dt) - \cos t\cdot(\cos t\,dt)$$

$\vec{g}(t) = \begin{pmatrix}\cos t\\ \sin t\\ 1\end{pmatrix}$

$$= -\int_0^{2\pi} dt = -2\pi.$$

Next, $d\omega = dy\wedge dx - dx\wedge dy + z\,dy\wedge dz$

$$= -2\,dx\wedge dy + z\,dy\wedge dz,$$

So

$$\int_S d\omega = \int_S -2\,dx\wedge dy + z\,dy\wedge dz$$

$$\Big\}\ G^*$$

$$\vec G\begin{pmatrix} r\\ \theta\end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2\end{pmatrix}$$

$$= \int_{[0,1]\times[0,2\pi]} \left(-2\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta\end{vmatrix} + r^2 \begin{vmatrix} y_r & y_\theta\\ z_r & z_\theta\end{vmatrix}\right) dr\wedge d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(-2\cdot r + r^2\cdot(-2r^2\cos\theta)\right) dr\,d\theta$$

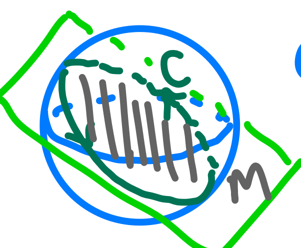
$$= -\int_0^{2\pi} \left(1+\tfrac{2}{5}\cos\theta\right) d\theta$$

$$= -\left[\theta + \tfrac{2}{5}\sin\theta\right]_0^{2\pi} = -2\pi.$$ ✓ //

One more:

Ex 4/ Let C be the intersection of $x+2y+z=0$ with $x^2+y^2+z^2=1$. Calculate $\oint_C \omega$, where

$$\omega = (z-x)\,dx + (x-y)\,dy + (y+e^{z^7})\,dz.$$



(Generalized) Stokes's Theorem

If $M\subset\mathbb{R}^n$ is a compact, oriented k-manifold with boundary ∂M, and ω is a $(k-1)$-form on M, then

$$\int_M d\omega = \int_{\partial M}\omega.$$

Taking $M = \{x+2y+z=0\}\cap\{x^2+y^2+z^2\le 1\}$,

$$\oint_C \omega = \int_M d\omega$$

$$= \int_M 1\,dy\wedge dz + 1\,dz\wedge dx + 1\,dx\wedge dy$$

$$= \int_M \vec F\cdot\hat n\,dS \qquad \vec F = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \int_M \frac{4}{\sqrt6}\,dS \qquad \hat n = \frac{1}{\sqrt6}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$= \frac{4}{\sqrt6}\,\text{area}(M)$$

but M is a unit disk!

$$= \frac{4}{\sqrt6}\pi.$$ //