DIV, GRAD, & CURL

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§1. Vector fields & differential forms

In $\mathbb{R}^3$, there are some classical ways of writing Stokes's theorem in terms of vector fields. These involve the following operations:

- $\vec{\nabla} f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ GRADIENT: sends functions to vector fields
- $\operatorname{div} \vec{F} = \operatorname{div}\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ DIVERGENCE: sends vector fields to functions
  $$:= P_x + Q_y + R_z$$
- $\operatorname{curl} \vec{F} = \operatorname{curl}\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ CURL: sends vector fields to vector fields
  $$:= \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix}$$

To express these in terms of forms, notice that

- $df = f_x\,dx + f_y\,dy + f_z\,dz$
- $d(P\,dx + Q\,dy + R\,dz) =$
  $(R_y - Q_z)\,dy\wedge dz + (P_z - R_x)\,dz\wedge dx + (Q_x - P_y)\,dx\wedge dy$
- $d(P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy)$
  $= (P_x + Q_y + R_z)\,dx\wedge dy\wedge dz.$

So the exterior derivative encodes all 3 operations.

$$\eta_{\vec{F}} := P\,dx + Q\,dy + R\,dz$$

so that also

$$*\eta_{\vec{F}} = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

Then

- $df = \eta_{\vec{\nabla} f}$,
- $*d\eta_{\vec{F}} = \eta_{\mathrm{curl}(\vec{F})}$, and
- $*d*\eta_{\vec{F}} = \mathrm{div}(\vec{F})$.

Since $d \circ d$ is always zero, we get

$$0 = d(df) = d\eta_{\vec{\nabla} f} = *\eta_{\mathrm{curl}(\vec{\nabla} f)}$$

$$\Longrightarrow \quad \mathrm{curl}(\vec{\nabla} f) = \vec{0} \qquad \text{and}$$

$$0 = d(d\eta_{\vec{F}}) = d*\eta_{\mathrm{curl}(\vec{F})} = *\,\mathrm{div}(\mathrm{curl}\,\vec{F})$$

$$\Longrightarrow \quad \mathrm{div}(\mathrm{curl}(\vec{F})) = 0.$$

What about other combinations of the operations, like $\mathrm{curl}(\mathrm{curl}(\vec{F}))$?

$\eta_{\vec{F}} := P\,dx + Q\,dy + R\,dz$

so that also

$*\eta_{\vec{F}} = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$

Then

- $df = \eta_{\vec{\nabla} f}$,
- $*d\eta_{\vec{F}} = \eta_{\mathrm{curl}(\vec{F})}$, and
- $*d*\eta_{\vec{F}} = \mathrm{div}(\vec{F})$.

Since $d \circ d$ is always zero, we get

$0 = d(df) = d\eta_{\vec{\nabla} f} = *\eta_{\mathrm{curl}(\vec{\nabla} f)}$

$\Rightarrow \quad \mathrm{curl}(\vec{\nabla} f) = \vec{0}$ and

$0 = d(d\eta_{\vec{F}}) = d*\eta_{\mathrm{curl}(\vec{F})} = *\,\mathrm{div}(\mathrm{curl}\,\vec{F})$

$\Rightarrow \quad \mathrm{div}(\mathrm{curl}(\vec{F})) = 0.$

What about other combinations of the operations, like $\mathrm{curl}(\mathrm{curl}(\vec{F}))$?

- $\eta_{\mathrm{curl}(\mathrm{curl}\,\vec{F})} = *d\eta_{\mathrm{curl}\,\vec{F}}$
$= *d*d\eta_{\vec{F}}$

and

$\eta_{\vec{\nabla}(\mathrm{div}\,\vec{F})} = d(\mathrm{div}\,\vec{F})$
$= d*d*\eta_{\vec{F}}$

<u>PROBLEM</u>: Check that (on 1-forms)

$d*d* - *d*d = \underbrace{\nabla^2 := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}}_{\text{Laplacian}}.$

Conclude that

$\vec{\nabla}(\mathrm{div}\,\vec{F}) - \mathrm{curl}(\mathrm{curl}\,\vec{F}) = \nabla^2 \vec{F}.$

$$\eta_{\vec F} := P\,dx + Q\,dy + R\,dz$$

so that also

$$*\eta_{\vec F} = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

Then

- $df = \eta_{\vec\nabla f}$,
- $*d\eta_{\vec F} = \eta_{\mathrm{curl}(\vec F)}$, and
- $*d*\eta_{\vec F} = \mathrm{div}(\vec F)$.

Since $d\circ d$ is always zero, we get

$$0 = d(df) = d\eta_{\vec\nabla f} = *\,\eta_{\mathrm{curl}(\vec\nabla f)}$$

$$\Longrightarrow \quad \mathrm{curl}(\vec\nabla f) = \vec 0 \quad \text{and}$$

$$0 = d(d\eta_{\vec F}) = d*\eta_{\mathrm{curl}(\vec F)} = *\,\mathrm{div}(\mathrm{curl}\,\vec F)$$

$$\Longrightarrow \quad \mathrm{div}(\mathrm{curl}(\vec F)) = 0.$$

What about other combinations of the operations, like $\mathrm{curl}(\mathrm{curl}(\vec F))$?

- $$\eta_{\mathrm{curl}(\mathrm{curl}\,\vec F)} = *d\eta_{\mathrm{curl}\,\vec F} = *d*d\eta_{\vec F}$$

and

$$\eta_{\vec\nabla(\mathrm{div}\,\vec F)} = d(\mathrm{div}\,\vec F) = d*d*\eta_{\vec F}$$

PROBLEM: Check that (on 1-forms)

$$d*d* - *d*d = \underbrace{\nabla^2 := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}}_{\text{Laplacian}}.$$

Conclude that

$$\vec\nabla(\mathrm{div}\,\vec F) - \mathrm{curl}(\mathrm{curl}\,\vec F) = \nabla^2 \vec F.$$

We check it on $P\,dx$ for simplicity:

$$d*d*P\,dx - *d*d\,P\,dx$$

$$= d*d(P\,dy\wedge dz) - *d*(P_z\,dz\wedge dx - P_y\,dx\wedge dy)$$

$$= d*(P_x\,dx\wedge dy\wedge dz) - *d(P_z\,dy - P_y\,dz)$$

$$= d(P_x) - *(P_{zx}\,dx\wedge dy - P_{zz}\,dy\wedge dz - P_{yy}\,dy\wedge dz + P_{yx}\,dz\wedge dx)$$

$$= P_{xx}\,dx + \cancel{P_{xy}\,dy} + \cancel{P_{xz}\,dz} - \cancel{P_{zx}\,dz} + P_{zz}\,dx + P_{yy}\,dx - \cancel{P_{yx}\,dy} \quad ✓$$

$$\eta_{\vec F} := P\,dx + Q\,dy + R\,dz$$

so that also

$$*\eta_{\vec F} = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy.$$

Then

- $df = \eta_{\vec\nabla f}$,
- $*d\eta_{\vec F} = \eta_{\text{curl}(\vec F)}$, and
- $*d*\eta_{\vec F} = \text{div}(\vec F)$.

Since $d\circ d$ is always zero, we get

$$0 = d(df) = d\eta_{\vec\nabla f} = *\eta_{\text{curl}(\vec\nabla f)}$$

$\Longrightarrow$ $\text{curl}(\vec\nabla f) = \vec 0$ and

$$0 = d(d\eta_{\vec F}) = d*\eta_{\text{curl}(\vec F)} = *\,\text{div}(\text{curl}\,\vec F)$$

$\Longrightarrow$ $\text{div}(\text{curl}(\vec F)) = 0$.

What about other combinations of the operations, like $\text{curl}(\text{curl}(\vec F))$?

- $$\eta_{\text{curl}(\text{curl}\,\vec F)} = *d\eta_{\text{curl}\,\vec F} = *d*d\eta_{\vec F}$$

and

$$\eta_{\vec\nabla(\text{div}\,\vec F)} = d(\text{div}\,\vec F) = d*d*\eta_{\vec F}$$

PROBLEM: Check that (on 1-forms)

$$d*d* - *d*d = \underbrace{\nabla^2 := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}}_{\text{Laplacian}}.$$

Conclude that

$$\vec\nabla(\text{div}\,\vec F) - \text{curl}(\text{curl}\,\vec F) = \nabla^2\vec F.$$

One more:

- $$\text{div}(\vec\nabla f) = \text{div}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = f_{xx} + f_{yy} + f_{zz} = \nabla^2 f.$$

Now, what does all this have to do with Stokes's Theorem?

§2. Vector forms of Stokes's Theorem

Recall that the theorem reads

$$(*)\qquad \int_{\partial M}\omega = \int_M d\omega$$

where M is a k-manifold, and $\omega\in A^{k-1}(M)$.

If $M\subset\mathbb{R}^3$, then $k=1,2,$ or 3.

- for $M=\mathcal{C}$ a curve,

$$\underbrace{\int_C \eta_{\vec F}}_{P\,dx+Q\,dy+R\,dz} =: \int_C \vec F\cdot \underbrace{\hat T\, ds}_{\text{or } d\vec r} \qquad \text{"work"}$$

- for $M=\mathcal{S}$ a surface,

$$\int_{\mathcal{S}} \underbrace{*\eta_{\vec F}}_{P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy} = \int \vec F\cdot\hat n\, dS \qquad \text{"flux"}$$

- for M a solid,

$$\int_M \underbrace{*f}_{f\,dx\wedge dy\wedge dz} = \int_M f\,dV$$

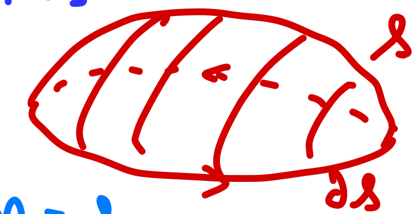
Stokes for $k=1$

(*) becomes $\qquad M=\mathcal{C},\ \omega=f$

$$f(\vec B)-f(\vec A) = \text{"}\int_{\partial \mathcal{C}} f\text{"} \boxed{\equiv} \int_{\mathcal{C}} df = \int_{\mathcal{C}} \eta_{\vec\nabla f}$$

$$= \int_{\mathcal{C}} \vec\nabla f\cdot\hat T\,ds$$

Stokes for $k=2$

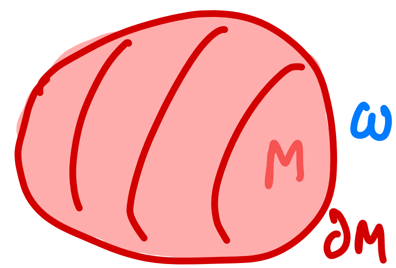


(*) reads $\qquad M=\mathcal{S},\ \omega=\eta_{\vec F}$

$$\oint_{\partial \mathcal{S}}\vec F\cdot\hat T\, ds = \int_{\partial \mathcal{S}}\eta_{\vec F} \boxed{\equiv}\int_{\mathcal{S}} d\eta_{\vec F} = \int_{\mathcal{S}} *\eta_{\mathrm{curl}(\vec F)}$$

$$= \int_{\mathcal{S}} \mathrm{curl}(\vec F)\cdot\hat n\, dS$$

Stokes for $k=3$



$\omega = *\eta_{\vec F}$

(*) translates to

$$\int_{\partial M}\vec F\cdot\hat n\, dS = \int_{\partial M} *\eta_{\vec F} \boxed{\equiv}\int_M d*\eta_{\vec F} = \int_M *\operatorname{div}\vec F$$

$$= \int_M \operatorname{div}(\vec F)\, dV.$$

In plain English:

- The change in the potential function of a conservative force field $\vec{F}$ from $\vec{A}$ to $\vec{B}$ is the total work done by $\vec{F}$ along $\mathcal{C}$.

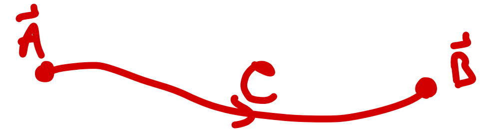
- The total circulation of $\vec{F}$ around $\partial \mathcal{S}$ is equal to the flux of curl $\vec{F}$ thru $\mathcal{S}$.

$$\left(\begin{array}{l} \text{If } \mathcal{S} \text{ is a very small disk, with radius } \epsilon \to 0, \text{ we can represent} \\ (\operatorname{curl} \vec{F}) \cdot \hat{n} = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{\partial D_\epsilon} \vec{F} \cdot \hat{T} \, ds \end{array} \right)$$

- The total flux of $\vec{F}$ through ∂M is equal to the integral of its divergence over M. ["Gauss's Divergence Theorem"]

$$\left(\begin{array}{l} \text{If } M \text{ is a very small ball, with radius } \epsilon \to 0, \text{ we can represent} \\ \operatorname{div} \vec{F} = \lim_{\epsilon \to 0} \frac{1}{\frac{4}{3}\pi \epsilon^3} \int_{\partial B_\epsilon} \vec{F} \cdot \hat{n} \, dS. \end{array} \right)$$

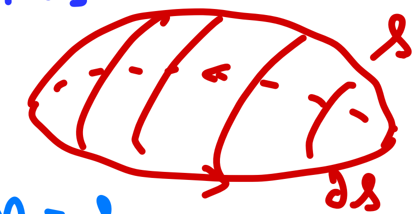
$\boxed{\text{Stokes for } k=1}$



(*) becomes $\quad M = \mathcal{C}, \ \omega = f$

$$f(\vec{B}) - f(\vec{A}) \text{ "}=\text{" } \int_{\partial \mathcal{C}} f \ \boxed{=} \int_{\mathcal{C}} df = \int_{\mathcal{C}} \eta_{\vec{\nabla} f}$$
$$= \int_{\mathcal{C}} \vec{\nabla} f \cdot \hat{T} \, ds$$

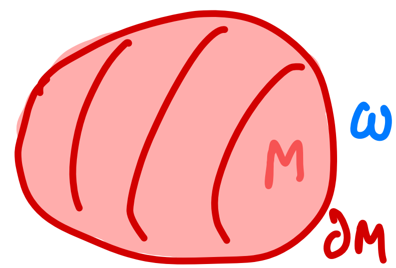
$\boxed{\text{Stokes for } k=2}$



(*) reads $\quad M = \mathcal{S}, \ \omega = \eta_{\vec{F}}$

$$\oint_{\partial \mathcal{S}} \vec{F} \cdot \hat{T} \, ds = \int_{\partial \mathcal{S}} \eta_{\vec{F}} \ \boxed{=} \int_{\mathcal{S}} d\eta_{\vec{F}} = \int_{\mathcal{S}} * \eta_{\operatorname{curl}(\vec{F})}$$
$$= \int_{\mathcal{S}} \operatorname{curl}(\vec{F}) \cdot \hat{n} \, dS$$

$\boxed{\text{Stokes for } k=3}$

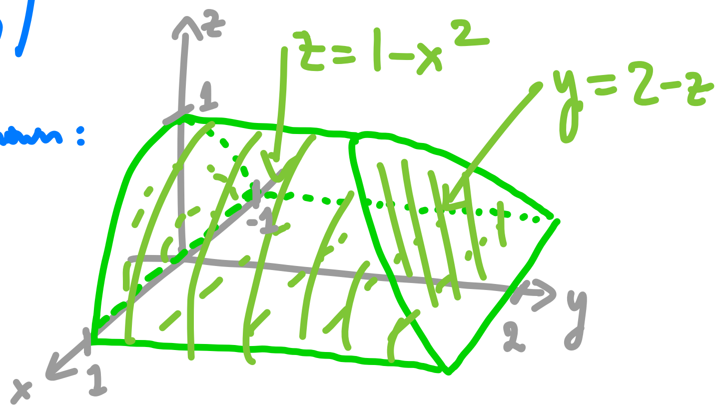


$\omega = * \eta_{\vec{F}}$

(*) translates to

$$\boxed{\int_{\partial M} \vec{F} \cdot \hat{n} \, dS} = \int_{\partial M} * \eta_{\vec{F}} \ \boxed{=} \int_M d * \eta_{\vec{F}} = \int_M * \operatorname{div} \vec{F}$$
$$= \boxed{\int_M \operatorname{div}(\vec{F}) \, dV}.$$

PROBLEM: Evaluate $\int_{S} \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = \begin{pmatrix} xy \\ y^2+e^{xz^2} \\ \sin(xy) \end{pmatrix}$ and S is the closed surface shown:

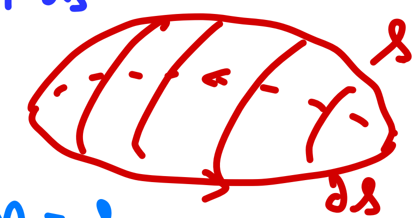


Stokes for $k=1$

(*) becomes $\qquad M = C,\ \omega = f$

$$\text{“}f(\vec{B})-f(\vec{A}) = \int_{\partial C} f\text{”} \boxed{\equiv} \int_C df = \int_C \eta_{\vec{\nabla} f}$$
$$= \int_C \vec{\nabla} f\cdot\hat{T}\,ds$$

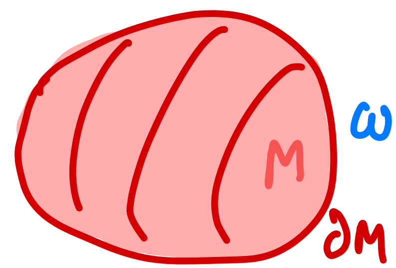
Stokes for $k=2$



(*) reads $\qquad M = S,\ \omega = \eta_{\vec{F}}$

$$\oint_{\partial S} \vec{F}\cdot\hat{T}\,ds = \int_{\partial S} \eta_{\vec{F}} \boxed{\equiv} \int_S d\eta_{\vec{F}} = \int_S * \eta_{\mathrm{curl}(\vec{F})}$$
$$= \int_S \mathrm{curl}(\vec{F})\cdot\hat{n}\,dS$$

Stokes for $k=3$

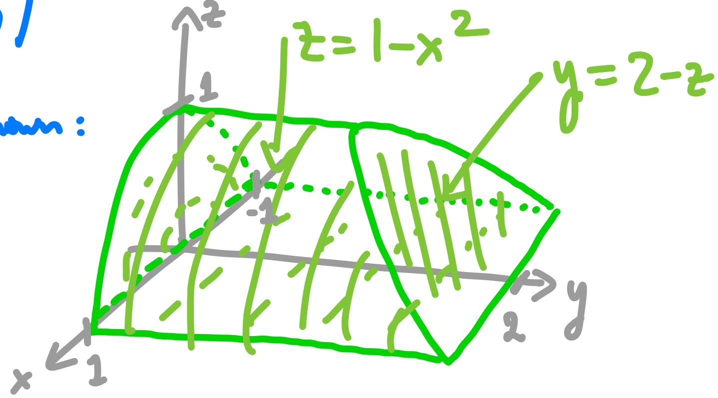


$\omega = *\eta_{\vec{F}}$

(*) translates to

$$\int_{\partial M} \vec{F}\cdot\hat{n}\,dS = \int_{\partial M} *\eta_{\vec{F}} \boxed{\equiv} \int_M d*\eta_{\vec{F}} = \int_M * \operatorname{div}\vec{F}$$
$$= \int_M \operatorname{div}(\vec{F})\,dV.$$

PROBLEM: Evaluate $\int_S \vec{F}\cdot\hat{n}\, dS$, where $\vec{F} = \begin{pmatrix} xy \\ y^2+e^{xz^2} \\ \sin(xy) \end{pmatrix}$ and S is the closed surface shown:



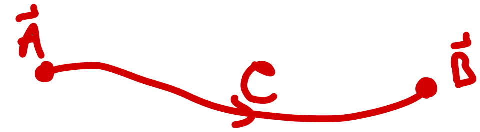
$S = \partial M$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2+e^{xz^2}\right) + \frac{\partial}{\partial z}\sin(xy)$$
$$= y + 2y + 0 = 3y.$$

By Gauss's divergence theorem (i.e. Stokes),

$$\int_S \vec{F}\cdot\hat{n}\, dS = \int_M \operatorname{div}(\vec{F})\, dV = \int_M 3y\, dV$$
$$= \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\, dy\, dz\, dx$$
$$= \dots \text{ [computation]}$$
$$= \frac{184}{25}.$$

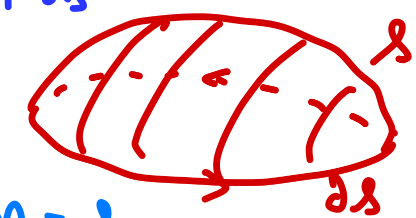
Stokes for $k=1$



(*) becomes $\quad M = C,\ \omega = f$

$$f(\vec{B}) - f(\vec{A}) = \text{“}\int_{\partial C} f\text{”} \boxed{=} \int_C df = \int_C \eta_{\vec{\nabla} f}$$
$$= \int_C \vec{\nabla} f \cdot \hat{T}\, ds$$

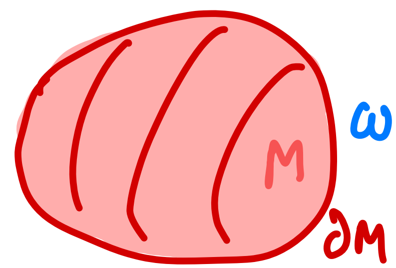
Stokes for $k=2$



(*) reads $\quad M = S,\ \omega = \eta_{\vec{F}}$

$$\oint_{\partial S} \vec{F}\cdot\hat{T}\, ds = \int_{\partial S} \eta_{\vec{F}} \boxed{=} \int_S d\eta_{\vec{F}} = \int_S *\eta_{\operatorname{curl}(\vec{F})}$$
$$= \int_S \operatorname{curl}(\vec{F})\cdot\hat{n}\, dS$$

Stokes for $k=3$



$\omega = *\eta_{\vec{F}}$

(*) translates to

$$\int_{\partial M} \vec{F}\cdot\hat{n}\, dS = \int_{\partial M} *\eta_{\vec{F}} \boxed{=} \int_M d{*\eta_{\vec{F}}} = \int_M *\operatorname{div}\vec{F}$$
$$= \int_M \operatorname{div}(\vec{F})\, dV.$$

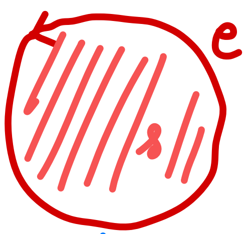
§3. Maxwell's equations (1st of 3 quick applications)

These describe the relationship between the electric & magnetic fields $\vec{E}$ & $\vec{B}$ (which depend on $\vec{x}$ and t): in a vacuum, the equations take the form

$$\begin{cases} \operatorname{div} \vec{E} = 0 = \operatorname{div} \vec{B} \\ \operatorname{curl} \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}, \quad \operatorname{curl} \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}. \end{cases}$$

They explain, for example, why a current in a coil (a) can move a magnet [solenoid] and (b) encounters "magnetic inertia" opposing any change to the electric current [inductor].

Applying Stokes's Theorem to the 3rd equation, for instance, gives that



$$\underbrace{\oint_C \vec{E}\cdot\hat{T}\, ds}_{\text{circulation of electric field around } C} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, dS$$

$$= \underbrace{-\frac{\partial}{\partial t}\int_S \vec{B}\cdot\hat{n}\, dS}_{\text{rate of change in magnetic flux across } S}$$

Now using the identity from §1,

$$\nabla^2 \vec{E} = \vec{\nabla}(\cancelto{0}{\operatorname{div}\vec{E}}) - \operatorname{curl}(\underbrace{\operatorname{curl}\vec{E}}_{-\partial\vec{B}/\partial t})$$

$$= \operatorname{curl}\left(\frac{\partial \vec{B}}{\partial t}\right) = \frac{\partial}{\partial t}\operatorname{curl}(\vec{B})$$

$$= \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

Similarly, $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{B}}{\partial t^2}$. These are exactly the PDEs for a wave in $\mathbb{R}^3$ propagating with speed $\dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Since the experimentally observed values of μ_0 & ε_0 and $c :=$ speed of light were such that $\dfrac{1}{\sqrt{\mu_0\varepsilon_0}} \sim c$, Maxwell hypothesized that light was an electromagnetic wave.

We also see from $\nabla^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2}$ that if $\vec{E}$ is not changing in time, then $\nabla^2 \vec{E} = 0$ — i.e. it is harmonic. In electrostatics this allows you to solve for an electric field with given boundary constraints.

§4. Gauss's law

Let $\delta(\vec{x})$ be a mass density function in $\mathbb{R}^3$. What is the gravitational field?

First, if we have a test mass M at $\vec{a}$

$$\vec{F}(\vec{x}) = -GM \frac{\vec{x}-\vec{a}}{\|\vec{x}-\vec{a}\|^3},$$

and we showed last lecture that for any closed surface S enclosing $\vec{a}$

$$\int_S \vec{F}\cdot\hat{n}\, dS = -4\pi G M.$$

It is also true that if S does <u>not</u> enclose $\vec{a}$, then $S=\partial\Omega$ with $\vec{a}\notin\Omega$ and Gauss's divergence theorem $\Rightarrow$

$$\int_S \vec{F}\cdot\hat{n}\, dS = \int_\Omega \cancel{\operatorname{div}\vec{F}}^{\,0}\, dV = 0.$$

So, if we have several masses M_i at $\vec{a}_i$ $(i=1,\dots,n)$,

$$\vec{F}(\vec{x}) = -G\sum_{i=1}^{n} M_i \frac{\vec{x}-\vec{a}_i}{\|\vec{x}-\vec{a}_i\|^3}$$

and if $S=\partial\Omega$

$$\int_S \vec{F}\cdot\hat{n}\, dS = -4\pi G \sum_{i:\,\vec{a}_i\in\Omega} M_i.$$

Finally, if there is a continuous mass distribution δ on a region D, the sum becomes an integral:

$$\vec{F}(\vec{x}) = -G\int_D \delta(\vec{y}) \frac{\vec{x}-\vec{y}}{\|\vec{x}-\vec{y}\|^3}\, dV.$$

Your job in the HW is to prove that, for S enclosing D, and M the total mass of D,

$$\int_S \vec{F}\cdot\hat{n}\, dS = -4\pi G M.$$

You will do this by plugging (*) into the left-hand side of (**) and changing the order of integration.

If S doesn't completely enclose D, then (**) holds but with $M :=$ the mass S does enclose. So for example if D is the Earth of radius R, and S a sphere of radius r, then:

$$\int_S \underbrace{\vec{F}\cdot\hat{n}}_{\text{parallel}}\, dS = -4\pi G \underbrace{\left(\frac{r}{R}\right)^3}_{\substack{\text{ratio of mass}\\ \text{enclosed by } S\\ \text{to mass of earth}}} M_{earth}$$

$$\| $$

$$-\left\|\vec{F}\begin{pmatrix}0\\0\\r\end{pmatrix}\right\| \cdot 4\pi r^2$$

$\Rightarrow$ force of gravity at a point on S is

$\left\|\vec{F}\begin{pmatrix}0\\0\\r\end{pmatrix}\right\| = \dfrac{G M_{earth}}{R^3}\, r$, i.e., depends linearly on r.

So, if we have several masses M_i at $\vec{a}_i$ $(i = 1, \dots, n)$,

$$\vec{F}(\vec{x}) = -G\sum_{i=1}^{n} M_i \frac{\vec{x}-\vec{a}_i}{\|\vec{x}-\vec{a}_i\|^3}$$

and if $S = \partial\Omega$

$$\int_S \vec{F}\cdot\hat{n}\, dS = -4\pi G \sum_{\substack{i:\\ \vec{a}_i\in\Omega}} M_i .$$

Finally, if there is a continuous mass distribution δ on a region D, the sum becomes an integral:

$$(*) \quad \vec{F}(\vec{x}) = -G\int_D \delta(\vec{y})\, \frac{\vec{x}-\vec{y}}{\|\vec{x}-\vec{y}\|^3}\, dV .$$

Your job in the HW is to prove that, for S enclosing D, and M the total mass of D,

$$(**) \quad \int_S \vec{F}\cdot\hat{n}\, dS = -4\pi G M .$$

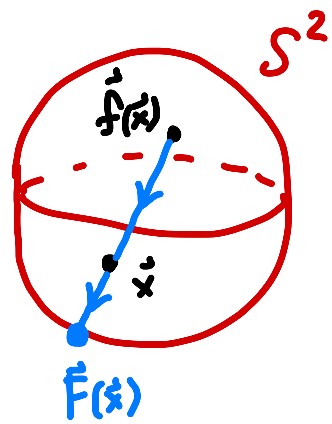
§5. A fixed-point theorem

Let

$$D^3 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| \leq 1\}$$

$$S^2 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}\| = 1\},$$

so that $\partial D^3 = S^2$.



We have the area form

$$\sigma = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

in $A^2(S^2)$, whose integral

$$\int_{S^2} \sigma = 4\pi. \qquad (\dagger)$$

Theorem: If $\vec{f} : D^3 \to D^3$ is C^∞, then it has a fixed point: i.e. $\vec{f}(\vec{a}) = \vec{a}$ for some $\vec{a} \in D^3$.

Proof: If $\vec{f}$ had no fixed point, then we can get a C^∞ map $\vec{F} : D^3 \to S^2$ as shown in the picture. Clearly if $\vec{x} \in S^2$, then $\vec{F}(\vec{x}) = \vec{x}$; so $\vec{F}|_{S^2} = id_{S^2}$.

Hence

$$\int_{S^2} \sigma = \int_{\vec{F}(S^2)} \sigma = \int_{S^2} \vec{F}^* \sigma$$

$$= \int_{\partial D^3} \vec{F}^* \sigma \underset{\text{Stokes}}{\boxed{=}} \int_{D^3} d\vec{F}^* \sigma$$

$$= \int_{D^3} F^* \underbrace{d\sigma} = 0$$

as a form on S^2, this is zero, because $A^3(S^2) = 0$.

But this contradicts $(\dagger)$. $\square$