APPLICATIONS + FINAL REVIEW

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§1. Homotopic maps

These are maps which can be smoothly deformed into each other. This has a powerful connection with integrals of differential forms.

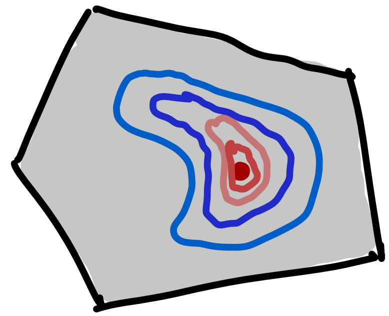
Definition: Let $M \subset \mathbb{R}^m$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two C^∞ maps $\vec{f}, \vec{g} : M \to X$ are *homotopic* (written $\vec{f} \simeq \vec{g}$) if there exists a C^∞ map

$$\vec{H} : M \times [0,1] \to X$$

with
$$\begin{cases} \vec{H}(\vec{p}, 0) = \vec{f}(\vec{p}) \\ \vec{H}(\vec{p}, 1) = \vec{g}(\vec{p}). \end{cases}$$

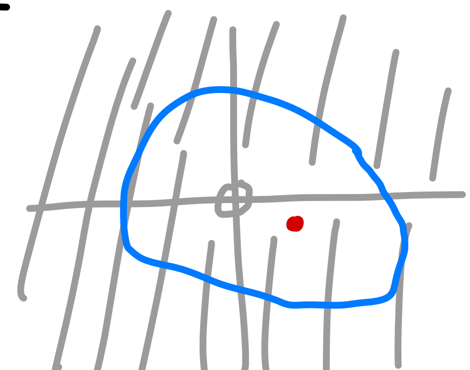
Ex/ If X is convex, any two maps into it are homotopic: put

$$\vec{H}(\vec{p}, t) := t\,\vec{g}(\vec{p}) + (1-t)\,\vec{f}(\vec{p}).$$



e.g., any map from S^1 into a convex X can be shrunk to a constant map (point).

We cannot do this for maps from S^1 into (say) $X = \mathbb{R}^2 \setminus \{\vec{0}\}$, because $\vec{H}$ has to stay inside X and the loop may not be moved across the origin:



Definition: Let $M \subset \mathbb{R}^m$ be a compact, oriented k-manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two C^∞ maps $\vec{f}, \vec{g} : M \to X$ are homotopic (written $\vec{f} \simeq \vec{g}$) if there exists a C^∞ map

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$$\begin{cases} \vec{H}(\vec{p}, 0) = \vec{f}(\vec{p}) \\ \vec{H}(\vec{p}, 1) = \vec{g}(\vec{p}). \end{cases}$$

Indeed, a map $\vec{r} : S^1 \to \mathbb{R}^2 \setminus \{0\}$ has a winding number (about $\vec{0}$) given by $\frac{1}{2\pi} \int_{S^1} \vec{r}^*\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right) \in \mathbb{Z}$. If $\vec{r}$ changes smoothly (without its image passing through $\vec{0}$) this must change continuously (in $\mathbb{Z}$!) so remains constant.

That is, "loops in $\mathbb{R}^2 \setminus \{0\}$ with different winding #'s are not homotopic". This idea is generalized by the following

Proposition: If $\vec{f} \simeq \vec{g}$, and $\omega \in A^k(X)$ is closed ($d\omega = 0$), then
$$\int_M \vec{f}^*\omega = \int_M \vec{g}^*\omega.$$

Proof: $0 = \int_{M \times [0,1]} \vec{H}^*(d\omega) = \int_{M \times [0,1]} d(\vec{H}^*\omega)$

$\overset{\text{Stokes}}{=} \int_{\partial(M \times [0,1])} \vec{H}^*\omega = \int_{M \times \{1\}} \vec{H}^*\omega - \int_{M \times \{0\}} \vec{H}^*\omega$

$= \int_M \vec{g}^*\omega - \int_M \vec{f}^*\omega.$ $\square$

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Proposition: If $\vec{f} \simeq \vec{g}$, and $\omega \in A^k(X)$ is closed ($d\omega = 0$), then

$$\int_M \vec{f}^* \omega = \int_M \vec{g}^* \omega.$$

Corollary: If X is simply connected (i.e. any $\vec{f} : S^1 \to X$ is homotopic to a constant map), then every closed 1-form on X is exact.

Proof: Suppose $\omega \in A^1(X)$, with $d\omega = 0$. For any closed curve C on X, the parametrizing map $\vec{f} : S^1 \to X$ is homotopic to a constant map $\vec{g}$. Since $g^*\omega = 0$, the Proposition gives

$$\oint_C \omega = \int_{S^1} \vec{f}^* \omega = \int_{S^1} \vec{g}^* \omega = 0.$$

It follows that $\omega = df$ for some function f on X. $\square$

We turn to two applications, one geometric & one algebraic.

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Proposition: If $\vec{f} \simeq \vec{g}$, and $\omega \in A^k(X)$ is closed ($d\omega = 0$), then

$$\int_M \vec{f}^* \omega = \int_M \vec{g}^* \omega.$$

§2. Why you shouldn't become a hairstylist to a sphere

A C^∞ vector field on the unit sphere S^2 is a (C^∞) map $\vec{v} : S^2 \to \mathbb{R}^3$ such that $\vec{v}(\vec{x}) \perp \vec{x}$ for each $\vec{x} \in S^2$. (That is, $\vec{v}(\vec{x})$ lies in the tangent plane $T_{\vec{x}} S^2$.)

Hairy sphere theorem: Any C^∞ vector field on S^2 must vanish somewhere.

Proof: Suppose $\vec{v}$ has no zero on S^2. Then we can define $\hat{v}(\vec{x}) := \dfrac{\vec{v}(\vec{x})}{\|\vec{v}(\vec{x})\|}$, hence a map $\hat{v} : S^2 \to S^2$.

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$$\int_M \vec{f}^*\omega = \int_M \vec{g}^*\omega.$$

Hairy sphere theorem: Any C^∞ vector field on S^2 must vanish somewhere.

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Let $\vec{f} : S^2 \to S^2$ be the identity
$\vec{x} \mapsto \vec{x}$
$\vec{g} : S^2 \to S^2$ the antipodal map
$\vec{x} \mapsto -\vec{x}$

and set

$$\vec{H}(\vec{x}, t) := (\cos \pi t)\,\vec{x} + (\sin \pi t)\,\hat{v}(\vec{x}).$$

This has $\vec{H}(\vec{x}, 0) = \vec{x} = \vec{f}(\vec{x})$, $\vec{H}(\vec{x}, 1) = -\vec{x} = \vec{g}(\vec{x})$, and $\|\vec{H}(\vec{x}, t)\|^2 = (\cos \pi t)^2 \underbrace{\|\vec{x}\|^2}_{1} + (\sin \pi t)^2 \underbrace{\|\hat{v}(\vec{x})\|^2}_{1} = 1$ by the Pythagorean thm. ($\Rightarrow \vec{H}(\vec{x}, t) \in S^2$).

Applying the Proposition to the area form $(\omega =)\,\sigma = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$,

$$4\pi = \int_{S^2} \sigma = \int_{S^2} \vec{f}^*\sigma \boxed{=} \int_{S^2} \vec{g}^*\sigma = \int_{S^2} -\sigma = -4\pi$$

※ □

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$$\int_M \vec{f}^* \omega = \int_M \vec{g}^* \omega.$$

§3. Why every polynomial has a zero

Let $p(z) = z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0$ be a polynomial with $\alpha_i \in \mathbb{C}$, in a complex variable z. (We shall identify $\mathbb{C}$ with $\mathbb{R}^2$, via $z = x + iy \Leftrightarrow \binom{x}{y}$.)

FUNDAMENTAL THEOREM OF ALGEBRA:

p has a root in $\mathbb{C}$.

Proof: Take $R > 0$ sufficiently large that $\left| \frac{p(z) - z^n}{z^n} \right| = \left| \frac{\alpha_{n-1}}{z} + \dots + \frac{\alpha_0}{z^n} \right| \leq \frac{1}{2}$.

Write $B := \{ z \in \mathbb{C} \mid |z| \leq R \}$. (large disk)

Define $H : \partial B \times [0,1] \to \mathbb{C} \setminus \{0\}$ by $H(z,t) := t z^n + (1-t)\, p(z)$, giving a homotopy between $g(z) = z^n$ & $p(z)$.

Proposition: If $\vec{f} \simeq \vec{g}$, and $\omega \in A^k(X)$ is closed ($d\omega = 0$), then
$$\int_M \vec{f}^*\omega = \int_M \vec{g}^*\omega.$$

Now $\omega = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ is closed on $\mathbb{R}^2\setminus\{\vec{0}\} = \mathbb{C}\setminus\{0\}$

and $g(z) = z^n$ for $z \in \partial B$ translates (think $z = Re^{it}$) to $\vec{g}\begin{pmatrix} R\cos t \\ R\sin t\end{pmatrix} = \begin{pmatrix} R^n\cos(nt) \\ R^n\sin(nt)\end{pmatrix}$.

So
$$\int_{\partial B} \vec{g}^*\omega = \int_0^{2\pi} (n\sin^2 nt + n\cos^2 nt)\,dt = 2\pi n$$
$\|$ Prop.
$$\int_{\partial B} \vec{p}^*\omega \underset{\text{Stokes}}{=} \int_B d\,p^*\omega = \int_B p^*\cancel{d\omega}^{\,0} = 0$$
※

and if p had no root in B, then p maps $B \to \mathbb{C}\setminus\{0\}$ and Stokes theorem applies, yielding the contradiction shown. ∎

§3. Why every polynomial has a zero

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[To check $\vec{H}$ doesn't hit $\vec{0}$, write
$$|\vec{H}(z,t)| = |z^n + (1-t)(p(z) - z^n)| \geq |z^n| - (1-t)|p(z)-z^n|$$
$$\geq |z^n| - (1-t)\frac{|z^n|}{2} = \left(\frac{1}{2} + \frac{t}{2}\right)R^n \geq \frac{R^n}{2} > 0.]$$

§4. The Exam

"Units" in this course:

PART I: Vectors, functions, & derivatives

Shifrin Ch. 1–3 (Midterm 1)

PART II: Matrix algebra, optimization, manifolds

Shifrin Ch. 4–6 (Midterm 2)

PART III: Multiple integrals

Shifrin Ch. 7

PART IV: Integration on manifolds

Shifrin Ch. 8 (thru §8.6)

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could also be
3 pp. on part IV
1 page on part III

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- matrix of a linear transformation
- dot product + projections
- cross product + scalar triple product
- topology in $\mathbb{R}^n$ (open & closed sets, boundary, interior, etc.)
- limits of sequences, Cauchy sequences, monotonic sequence theorem
- limits of functions, continuity
- partial and directional derivatives, gradients (normal to level sets, use in defining their tangent planes)
- differentiability of functions of several variables
- Jacobians (i.e. $D\vec{F}$ as matrix) and the chain rule $(D\vec{F} \circ D\vec{G} = D(\vec{F} \circ \vec{G}))$
- higher partials, Clairaut's theorem
- parametric curves, arclength, Kepler's laws, curvature

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- dictionary between linear systems & augmented matrices, EROs
- using RREF to solve systems, determine independence of vectors, ranks of matrices, and compute inverses
- rank + nullity for transformations, finding bases for $\mathrm{Col}(A)$ / $\mathrm{Im}(T)$ & $\mathrm{Nul}(A)$ / $\mathrm{Ker}(T)$ (& $\mathrm{Row}(A)$)
- implicit partial differentiation, implicit function theorem, & manifolds (key point: if M defined by $\vec{F} = \vec{c}$, then need $D\vec{F}$ of maximal rank everywhere)
- extrema of scalar-valued functions (local extrema occur at critical points, compactness guarantees global extrema)
- 2nd derivative test, Hessian matrices and quadratic forms (for determining when stationary pt. is an extremum)
- eigenvalues, eigenvectors, spectral theorem
- Lagrange multipliers
- projection formulas & least-squares

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- extrema of scalar-valued functions (local extrema occur at critical points, compactness guarantees global extrema)
- 2nd derivative test, Hessian matrices and quadratic forms (for determining when stationary pt. is an extremum)
- eigenvalues, eigenvectors, spectral theorem
- Lagrange multipliers
- projection formulas & least-squares, Gram-Schmidt, o.n. bases, orthogonal matrices
- Contraction mapping fixed point theorem, inverse & implicit function theorems, and "explicit, implicit, & parametric" characterizations of manifolds

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- definition of multiple integrals via partitions of rectangle, extension to nonrectangular regions; a bounded function on a region with discontinuity set of content zero is integrable
- Fubini's theorem in the rectangular & nonrectangular setting; switching the order of integration
- polar, cylindrical, & spherical integration
- applications: volume, center of mass, average value, moment of inertia, gravitational field
- determinants as alternating multilinear functionals, behavior under row/column operations; Laplace expansion, relation to invertibility, volume expansion factor
- change-of-variable formula for multiple integrals:

$$\int_{\vec{g}(\Omega)} \vec{F}(\vec{x})\, dV = \int_{\Omega} \vec{F}(\vec{g}(\vec{u}))\, |\det D\vec{g}|\, dV$$

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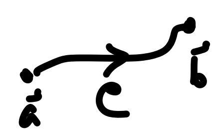
- differential forms: as alternating multilinear functions on k-tuples of vectors; sums & products, exterior derivative d
- closed ($d\omega = 0$) vs. exact ($\omega = d\eta$): exact $\Rightarrow$ closed in general; but closed $\Rightarrow$ exact depends on "topology" of region (no issues for $\mathbb{R}^n$, convex, etc.)
- pullbacks of forms under a smooth map: $\vec{g}: V \to U \rightsquigarrow \vec{g}^*: \mathcal{A}^k(U) \to \mathcal{A}^k(V)$ (commutes with d)
- integrating k-forms over k-manifolds: $\int_{\vec{g}(\Omega)} \omega := \int_\Omega \vec{g}^*\omega$ (change-of-variable is built in)
- line integrals $\int_C P\,dx + Q\,dy + \dots$, connection to work, $\int_C \vec{F}\cdot d\vec{r}$; independence of path $\Leftrightarrow$ $\omega = df$ exact: $\int_C df = f(\vec{b}) - f(\vec{a})$



- surface integrals: how to compute

$$\int_S f\,dS = \int f\,\sigma \quad \text{(integration w.r.t. surface area)}$$

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$$

(flux of $\vec{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ across S)

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  $\int_C df = f(\vec{b}) - f(\vec{a})$
- surface integrals: how to compute
  $\int_S f\,dS = \int f\,\sigma$ (integration w.r.t. surface area)
  $\int_S \vec{F}\cdot\hat{n}\,dS = \int P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$ (flux of $\vec{F} = \begin{pmatrix}P\\Q\\R\end{pmatrix}$ across S)
- Stokes's Theorem $\left(\int_{\partial M}\omega = \int_M d\omega\right)$

  and the important special cases:
  - Green's theorem in the plane
  - Classical Stokes's theorem
    $$\iint_S (\operatorname{curl}\vec{F})\cdot\hat{n}\,dS = \oint_{\partial S}\vec{F}\cdot d\vec{s}$$
  - Gauss's divergence theorem
    $$\iiint_V \operatorname{div}\vec{F}\,dV = \iint_{\partial V}\vec{F}\cdot\hat{n}\,dS$$

(See lecture 26 §2 – these are completely spelled out)

[entirely possible there may be 2 Stokes's thm problems on the exam]