Applications + Final Review
§1. Homotopic maps

These are maps which can be smoothly deformed into each other. This has a powerful connection with integrals of differential forms.

Definition: Let $M \subset \mathbb{R}^m$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two $C^\infty$ maps $\tilde{f}, \tilde{g} : M \rightarrow X$ are homotopic (written $\tilde{f} \approx \tilde{g}$) if there exists a $C^\infty$ map

$\tilde{H} : M \times [0,1] \rightarrow X$

with

$\begin{cases}
\tilde{H}(\tilde{p}, 0) = \tilde{f}(\tilde{p}) \\
\tilde{H}(\tilde{p}, 1) = \tilde{g}(\tilde{p})
\end{cases}$

Ex/ If $X$ is convex, any two maps into it are homotopic: put

$\tilde{H}(\tilde{p}, t) := t \tilde{g}(\tilde{p}) + (1-t) \tilde{f}(\tilde{p})$.

E.g., any map from $S^1$ into a convex $X$ can be shrunk to a constant map (point).

We cannot do this for maps from $S^1$ into (say) $X = \mathbb{R}^2 \setminus \{0\}$, because $\tilde{H}$ has to stay inside $X$ and the loop may not be moved across the origin.
Definition: Let $M \subset \mathbb{R}^n$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two $C^\infty$ maps $f, g : M \to X$ are homotopic (written $f \simeq g$) if there exists a $C^\infty$ map $H : M \times [0, 1] \to X$ with
\[
\begin{cases}
H(f, 0) = f, \\
H(g, 1) = g.
\end{cases}
\]

Indeed, a map $F : S^1 \to \mathbb{R}^2 \setminus \{0\}$ has a winding number (about 0) given by
\[
\frac{1}{2\pi} \int_{S^1} F^* \left( -\frac{y\, dx + x\, dy}{x^2 + y^2} \right) \in \mathbb{Z}.
\]
If $F$ changes smoothly (without its image passing through 0) this must change continuously (in $\mathbb{Z}$!) so remains constant.

That is, "loops in $\mathbb{R}^2 \setminus \{0\}$ with different winding numbers are not homotopic". This idea is generalized by the following proposition:

Proposition: If $f \simeq g$, and $\omega \in A^k(X)$ is closed ($d\omega = 0$), then
\[
\int_M f^* \omega = \int_M g^* \omega.
\]

Proof: $0 = \int_{M \times [0, 1]} \tilde{H}^* (d\omega) = \int_{M \times [0, 1]} d(\tilde{H}^* \omega)
= \int_{M \times [0, 1]} \tilde{H}^* \omega - \int_{M \times \{0\}} \tilde{H}^* \omega
= \int_{M} g^* \omega - \int_{M} f^* \omega.$ □
**Definition:** Let $M \subset \mathbb{R}^n$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two $C^k$ maps $\tilde{f}, \tilde{g} : M \to X$ are homotopic (written $\tilde{f} \simeq \tilde{g}$) if there exists a $C^k$ map

$\tilde{H} : M \times [0,1] \to X$

with

\[
\begin{align*}
\tilde{H}(\mu, 0) &= \tilde{f}(\mu) \\
\tilde{H}(\mu, 1) &= \tilde{g}(\mu).
\end{align*}
\]

**Proposition:** If $\tilde{f} \simeq \tilde{g}$, and $w \in A^k(X)$ is closed ($\partial w = 0$), then

$\int_M \tilde{f}^* w = \int_M \tilde{g}^* w.$

**Corollary:** If $X$ is simply connected (i.e. any $\tilde{f} : S^1 \to X$ is homotopic to a constant map), then every closed 1-form on $X$ is exact.

**Proof:** Suppose $w \in A^1(X)$, with $\partial w = 0$. For any closed curve $C$ on $X$, the parametrization map $\tilde{f} : S^1 \to X$ is homotopic to a constant map $\tilde{g}$. Since $\tilde{g}^* w = 0$, the Proposition gives

$\int_C \tilde{f}^* w = \int_{S^1} \tilde{f}^* w = \int_{S^1} \tilde{g}^* w = 0.$

It follows that $w = df$ for some function $f$ on $X$. □

We turn to two applications, one geometric & one algebraic.
2. Why you shouldn't become a hair stylist to a sphere

A $C^0$ vector field on the unit sphere $S^2$ is a $(C^0)$ map $\mathbf{v}: S^2 \to \mathbb{R}^3$ such that $\mathbf{v}(\bar{x}) \perp \bar{x}$ for each $\bar{x} \in S^2$. (That is, $\mathbf{v}(\bar{x})$ lies in the tangent plane $T_{\bar{x}}S^2$.)

**Hairy sphere theorem:** Any $C^0$ vector field on $S^2$ must vanish somewhere.

**Proof:** Suppose $\mathbf{v}$ has no zero on $S^2$. Then we can define $\hat{\mathbf{v}}(\bar{x}) := \frac{\mathbf{v}(\bar{x})}{\|\mathbf{v}(\bar{x})\|}$, hence a map $\hat{\mathbf{v}}: S^2 \to S^2$.

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**Definition:** Let $M \subset \mathbb{R}^m$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two $C^0$ maps $\tilde{f}, \tilde{g}: M \to X$ are homotopic (written $\tilde{f} \simeq \tilde{g}$) if there exists a $C^0$ map $\tilde{H}: M \times [0,1] \to X$

such that

$\tilde{H}(\tilde{f}, 0) = \tilde{f}(\tilde{x})$

$\tilde{H}(\tilde{f}, 1) = \tilde{g}(\tilde{x})$.

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**Proposition:** If $\tilde{f} \simeq \tilde{g}$, and $\omega \in \mathcal{A}^k(X)$
is closed ($d\omega = 0$), then

$\int_M \tilde{f}^* \omega = \int_M \tilde{g}^* \omega$. 

Definition: Let \( M \subset \mathbb{R}^m \) be a compact, oriented manifold with \( \partial M = \emptyset \), and \( X \subset \mathbb{R}^n \) any manifold. Two \( C^\infty \) maps \( \tilde{f}, \tilde{g}: M \to X \) are homotopic (written \( \tilde{f} \simeq \tilde{g} \)) if there exists a \( C^\infty \) map

\[
\tilde{H}: M \times [0, 1] \to X
\]

with

\[
\begin{cases}
\tilde{H}(\tilde{f}, 0) = \tilde{f}(	ilde{x}) \\
\tilde{H}(\tilde{f}, 1) = \tilde{g}(	ilde{x})
\end{cases}
\]

Hairy sphere theorem: Any \( C^\infty \) vector field on \( S^2 \) must vanish somewhere.

Proof: Suppose \( \tilde{v} \) has no zero on \( S^2 \). Then we can define \( \tilde{v}(\tilde{x}) = \frac{\tilde{x}(\tilde{x})}{||\tilde{v}(\tilde{x})||} \), hence a map \( \tilde{\nu}: S^2 \to S^2 \).

Let \( \tilde{f}: S^2 \to S^1 \) be the identity

\[
\tilde{g}: S^2 \to S^2 \text{ the antipodal map}
\]

and set

\[
\tilde{H}(\tilde{x}, \tilde{v}) := (\cos \tau \tilde{x}^2 + \sin \tau \tilde{v})
\]

This has \( \tilde{H}(\tilde{x}, 0) = \tilde{x} = \tilde{f}(\tilde{x}) \), \( \tilde{H}(\tilde{x}, 1) = -\tilde{x} = \tilde{g}(\tilde{x}) \), and

\[
||\tilde{H}(\tilde{x}, \tilde{v})||^2 = (\cos \tau \tilde{x}^2)(||\tilde{x}||^2 + (\sin \tau \tilde{v})^2)||\tilde{v}||^2 = 1
\]

by the Pythagorean theorem. (\( \Rightarrow \tilde{H}(\tilde{x}, \tilde{v}) \in S^2 \)).

Applying the Proposition to the area form

\( \omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \),

\[
4\pi = \int_{S^2} \omega = \int_{S^2} \tilde{f}^* \omega = \int_{S^2} \tilde{g}^* \omega = \int_{S^2} \omega = 4\pi
\]

\( \Box \)
Definition: Let $M \subset \mathbb{R}^n$ be a compact, oriented manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two $C^\infty$ maps $\hat{f}, \hat{g} : M \to X$ are homotopic (written $\hat{f} \simeq \hat{g}$) if there exists a $C^\infty$ map

$$
\tilde{H} : M \times [0, 1] \to X
$$

with

$$
\begin{cases}
\tilde{H}(\hat{f}, 0) = \hat{f}(\hat{p}) \\
\tilde{H}(\hat{f}, 1) = \hat{g}(\hat{p}).
\end{cases}
$$

Proposition: If $\hat{f} \simeq \hat{g}$, and $\omega \in \Lambda^k(X)$ is closed ($d\omega = 0$), then

$$
\int_M \hat{f}^* \omega = \int_M \hat{g}^* \omega.
$$

\section{Why every polynomial has a zero}

Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be a polynomial with $a_i \in \mathbb{C}$, in a complex variable $z$. (We shall identify $\mathbb{C}$ with $\mathbb{R}^2$, via $z = x + iy \leftrightarrow (x, y)$.)

\textbf{Fundamental Theorem of Algebra:}

$p$ has a root in $\mathbb{C}$.

Proof: Take $R > 0$ sufficiently large that

$$
\left| \frac{p(z) - z^n}{z^n} \right| = \left| \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \leq \frac{1}{2}.
$$

Write $B := \{ z \in \mathbb{C} \mid |z| \leq R \}$. (large disk)

Define $H : \partial B \times [0, 1] \to \mathbb{C}\setminus \{0\}$ by

$$
H(z, t) := t z^n + (1 - t) p(z),
$$

giving a homotopy between $g(z) = z^n \& p(z)$.
Proposition: If \( \tilde{f} \equiv \tilde{g} \), and \( \omega \in A^k(X) \) is closed (\( d\omega = 0 \)), then
\[
\int_M \tilde{f}^* \omega = \int_M \tilde{g}^* \omega.
\]

\[\text{Now } \omega = -\frac{y \, dx + x \, dy}{x^2 + y^2} \text{ is closed on } \mathbb{R}^2 \setminus \{(0,0)\} \text{ and } g(z) = z^n \text{ for } z \in B \text{ translates (think } z = R e^{i \theta}) \text{ to } \tilde{g}(R \cos \theta) = (R^n \cos(n \theta)) \text{.}
\]

So
\[
\int_B \tilde{g}^* \omega = \int_0^{2\pi} (n \sin^2 n \theta + n \cos^2 n \theta) \, d\theta = 2\pi n \]

\[\text{Prop.}
\]
\[
\int_B \tilde{p}^* \omega = \int_B \tilde{d} \tilde{p}^* \omega = \int_B \tilde{p}^* d \omega = 0 \]

and if \( p \) had no root in \( B \), then \( p \) maps \( B \to \mathbb{C} \setminus \{0\} \) and Stokes' theorem applies, yielding the contradiction shown. □

### Exercise 3.

Why every polynomial has a zero

Let \( p(z) = z^n + \ldots + a_1 z + a_0 \) be a polynomial with \( a_0 \in \mathbb{C} \), in a complex variable \( z \). (We shall identify \( \mathbb{C} \) with \( \mathbb{R}^2 \), via \( z = x + iy \Leftrightarrow (x, y) \).)

**Fundamental Theorem of Algebra:**

\( p \) has a root in \( \mathbb{C} \).

**Proof:** Take \( R > 0 \) sufficiently large that
\[
\left| \frac{p(z) - z^n}{z^n} \right| = \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \leq \frac{1}{2}.
\]

Write \( B := \{ z \in \mathbb{C} \mid |z| \leq R \} \). (large balls)

Define \( H : \partial B \times [0, 1] \to \mathbb{C} \setminus \{0\} \) by \( H(z, t) := t z^n + (1-t) p(z) \),

\[\text{giving a homotopy between } g(z) = z^n \text{ and } p(z).\]

[To check \( H \) doesn't hit 0, write
\[
|H(z, t)| = |z^n + (1-t) (p(z) - z^n)| \geq |z^n| - (1-t) |p(z) - z^n| \geq |z^n| - (1-t) \frac{|z^n|}{2} = \frac{(z + \frac{z}{2}) R^n \geq R^n > 0.}
\]
3.4. The Exam

"Units" in this course:

**Part I**: Vectors, functions, & derivatives
   Shifrin Ch. 1-3 (Midterm 1)

**Part II**: Matrix algebra, optimization, manifolds
   Shifrin Ch. 4-6 (Midterm 2)

**Part III**: Multiple integrals
   Shifrin Ch. 7

**Part IV**: Integration on manifolds
   Shifrin Ch. 8 (thm 38.6)

Exam will be 6 pp., with 1 page each on Parts I & II, and 2 pp. each on Parts III & IV. It will be distributed on Friday, May 7 no later than 10:30 PM & due Monday, May 10 by 10:30 PM.

Could also be:
3 pp. on part IV
1 page on part III
34. The Exam

"Units" in this course:

**PART I: Vectors, functions, & derivations**

Shifrin Ch. 1-3 (Midterm 1)

**PART II: Matrix algebra, optimization, manifolds**

Shifrin Ch. 4-6 (Midterm 2)

**PART III: Multiple integrals**

Shifrin Ch. 7

**PART IV: Integration on manifolds**

Shifrin Ch. 8 (thus § 8.6)

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- Matrix of a linear transformation
- Dot product & projections
- Cross product & scalar triple product
- Topology in $\mathbb{R}^n$ (open & closed sets, boundary, interior, etc.)
- Limits of sequences, Cauchy sequences, monotone sequence theorem
- Limits of functions, continuity
- Partial and directional derivatives, gradients (normal to level sets, use in defining their tangent planes)
- Differentiability of functions of several variables
- Jacobians (i.e. $DF$ as matrix) and the chain rule ($D(F \circ G) = D(F) \cdot G$)
- Higher partials, Clairaut's theorem
- Parametric curves, arclength, Kepler's laws, curvature
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Shifrin Ch. 8 (thm §8.6)

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- Elementary row operations (ESR), augmented matrices, EROs
- Using RREF to solve systems, determine independence of vectors, ranks of matrices, and compute inverses
- Rank & nullity for transformations, finding bases for Col(A) / Im(T) & Nul(A) / Ker(T) (and Row(A))
- Implicit partial differentiation, implicit function theorem, & manifolds (key point: if \( \mathbf{F} \) defined by \( \mathbf{F} = \overline{c} \), then has D\( \mathbf{F} \) of maximal rank everywhere)
- Extreme of scalar-valued function (local extrema occur at critical points, compactness guarantees global extrema)
- 2nd derivative test, Hessian matrices & quadratic forms (for determining when stationary pt. is an extremum)
- Eigenvalues, eigenvectors, spectral theorem
- Lagrange multipliers
- Projection formulas & least-squares
34. The Exam

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  - Shifrin Ch. 4-6 (Midterm 2)

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  - Shifrin Ch. 7

- **PART IV**: Integration on manifolds
  - Shifrin Ch. 8 (thru §8.6)

Exam will be 6 pp., with 1 page each on Parts I & II, and 2 pp. each on Parts III & IV. It will be distributed on Friday, May 7 no later than 10:30 PM & due Monday, May 10 by 10:30 PM.

- Implicit partial differentiation, implicit function theorem, & manifolds (Key point: if $M$ defined by $F = c$, then need $DF$ of maximal rank everywhere)
- Extrema of scalar-valued functions (local extrema occur at critical points, compactness guarantees global extrema)
- 2nd derivative test, Hessian matrices and quadratic forms (for determining when stationary pt. is an extremum)
- Eigenvalues, eigenvalues, Spectral Theorem
- Lagrange multipliers
- Projection formulas & least-squares, Gram-Schmidt, o.n. bases, orthogonal matrices
- Contraction mapping fixed point theorem, inverse & implicit function theorems, and "explicit, implicit, & parametric" characterization of manifolds
34. The Exam

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Shifrin Ch. 1-3 (Midterm 1)

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Shifrin Ch. 4-6 (Midterm 2)

**PART III:** Multiple integrals
Shifrin Ch. 7

**PART IV:** Integration on manifolds
Shifrin Ch. 8 (thm 38.6)

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- Definition of multiple integrals over partitions of rectangles, extension to non-rectangular regions; a bounded function on a region with discontinuity set of content zero is integrable
- Fubini's theorem in the rectangular & non-rectangular setting; switching the order of integration
- Polar, cylindrical, & spherical integration
- Applications: volume, center of mass, average value, moment of inertia, gravitational field
- Determinants as alternating multilinear functionals, behavior under row/column operations; Laplace expansion, relation to invertibility, volume expansion factor
- Change-of-variable formulas for multiple integrals:

\[
\int_{\mathcal{V}} F(x) \, dV = \int_{\mathcal{V}} F(g(x)) |\det Dg| \, dV
\]

\[
\int_{\mathcal{V}} F(x) \, dV = \int_{\mathcal{V}} F(g(y)) |\det Dg| \, dV
\]
34. The Exam

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Shifrin Ch. 8 (the 38.6)

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- differential forms: as alternating multilinear functions on k-tuples of vectors; sums & products, exterior derivative d
- closed (d\omega = 0) vs. exact (\omega = d\varphi): exact \implies cloud in general; but closed \implies exact depends on "topology" of region (no issues for IR^n, convex, etc.)
- pullbacks of forms under a smooth map: \tilde{g}: V \to U \to \tilde{g}^* : A^k(U) \to A^k(V) (commutes with d)
- integrating k-forms on k-manifolds: \int_M \omega := \int_M \tilde{g}^* \omega (change of variables is built in)
- Line integral \int_C \Phi dx + \Lambda dy + \ldots, connection to work, \int_C \tilde{F} \cdot d\tilde{S}
  - independence of path \iff \omega = df exact: \int_C df = f(b) - f(a)
- Surface integrals: how to compute
  \[ \int_S f\sigma = \int_S f \sigma \] (integrating w.r.t. surface area)
  \[ \int_S \tilde{F} \cdot n d\tilde{S} = \int Pd\gamma + Qd\delta + Rd\kappa \] (flux of \tilde{F} across \tilde{S})
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Shifrin Ch. 8 (thm §8.6)

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- Line integrals \( \int_C P \, dx + Q \, dy + \cdots \),
  connection to work, \( \int_C \mathbf{F} \cdot d\mathbf{r} \);
  independence of path \( \Leftrightarrow \mathbf{w} = df \) exact:
  \( \int_C df = f(c) - f(a) \)

- Surface Integrals: how to compute
  \( \int_S f \, dS = \iint_S f \, d\sigma \) (integrate w.r.t. surface area)
  \( \iint_S \mathbf{F} \cdot \hat{n} \, dS = \iint_D \mathbf{F} \cdot \mathbf{n} \, d\sigma \) (flux of \( \mathbf{F} = (\mathbf{\hat{a}}) \) across \( D \))

- Stokes's Theorem \( \int_M \text{w} = \int_M \text{d}w \)
  and the important special cases:
  - Gauss's theorem in the plane
  - Classical Stokes's theorem
    \( \iint_S (\text{curl} \, \mathbf{F}) \cdot \hat{n} \, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \)
  - Gauss's divergence theorem
    \( \iiint_V \text{div} \, \mathbf{F} \, dV = \iint_{\partial V} \mathbf{F} \cdot \hat{n} \, dS \)

(See lecture 26 §2 (there are completely spaced out))

[entirely possible there may be 2
Stokes's thm. problems on the exam]