LIMITS

3

The functions in whose limits, derivatives, and integrals we are interested in multivariable Calculus are sometimes defined on all of $\mathbb{R}^n$, but often their domain is a subset of $\mathbb{R}^n$.

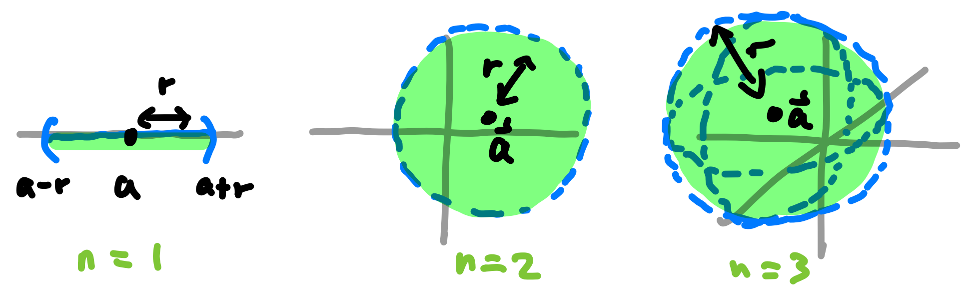
The nature of such subsets and their behavior under functions (even those defined on all of $\mathbb{R}^n$) is closely intertwined with the concepts of limits and continuity.

So here are some basic ideas about such sets.

§1. Subsets of $\mathbb{R}^n$

Given $\vec{a} \in \mathbb{R}^n$ and $r > 0$, define the open ball of radius r about $\vec{a}$ by

$$B(\vec{a}, r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < r\}.$$



Writing $\mathcal{S}^c := \mathbb{R}^n \setminus \mathcal{S}$ for the complement, define the boundary (or frontier) of $\mathcal{S}$ by

$$\partial \mathcal{S} := \mathbb{R}^n \setminus (\mathrm{int}(\mathcal{S}) \cup \mathrm{int}(\mathcal{S}^c))$$

$$= \{\vec{b} \in \mathbb{R}^n \mid B(\vec{b}, s) \cap \mathcal{S} \;\&\; B(\vec{b}, s) \cap \mathcal{S}^c \text{ are nonempty } \forall s > 0\}.$$

We say that $\mathcal{S}$ is open in $\mathbb{R}^n$ if $\mathcal{S} = \mathrm{int}(\mathcal{S})$, and closed if $\mathcal{S}^c$ is open.

i.e. $\mathcal{S}$ contains an open ball about each of its points

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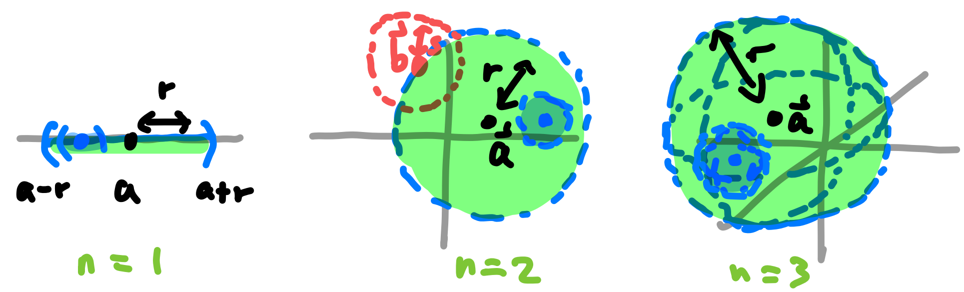


If $\mathcal{S} \subset \mathbb{R}^n$ is any subset, define its interior by

$$\mathrm{int}(\mathcal{S}) := \{\vec{a} \in \mathcal{S} \mid B(\vec{a}, r) \subset \mathcal{S} \text{ for some } r > 0\}.$$

Writing $S^c := \mathbb{R}^n \setminus S$ for the complement, define the boundary (or frontier) of S by

$$\partial S := \mathbb{R}^n \setminus (\operatorname{int}(S) \cup \operatorname{int}(S^c))$$

$$= \{\vec{b} \in \mathbb{R}^n \mid B(\vec{b},s) \cap S \ \& \ B(\vec{b},s) \cap S^c \text{ are nonempty } \forall s > 0\}.$$

We say that S is open in $\mathbb{R}^n$ if $S = \operatorname{int}(S)$, and closed if S^c is open. Alternatively, S is closed if it contains its boundary ∂S. For an arbitrary subset S, the closure is $\bar{S} := S \cup \partial S$, and this is the smallest closed set containing S (why?).

S closed $\iff$ S^c open
$\iff$ S^c contains an open ball about each of its points
$\iff$ no point of S^c belongs to ∂S

Exercise Which of the following subsets of $\mathbb{R}^n$ are open? closed? neither?

(1) $(0,2] \subset \mathbb{R}$

(2) $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y > 0\right\} \subset \mathbb{R}^2$

(3) $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, y = x\right\} \subset \mathbb{R}^2$

(4) $\left\{\vec{x} \in \mathbb{R}^n \,\middle|\, 0 < \|\vec{x}\| < 1\right\} \subset \mathbb{R}^n$

(5) $\mathbb{Q} \subset \mathbb{R}$ (the set of rational #'s)

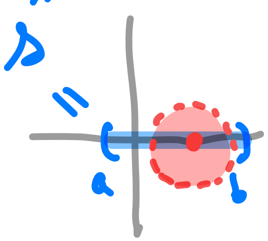
What do their closures and interiors look like?

Examples of open sets:

- open balls (includes open intervals (a,b))
- unions and Cartesian products of open sets, like $(a,b) \times (c,d)$
- $\mathbb{R}^n$ (as a subset of itself)
- $\emptyset$ (the empty set)

  — both open + closed
- a single point is closed; its complement is open.

Remark: Openness is "relative" — the interval (a,b) is open in $\mathbb{R}$, but if you include it into $\mathbb{R}^2$ as $\underbrace{(a,b) \times \{0\}}_{= S}$, it is not open in $\mathbb{R}^2$: there is not a 2-ball about each point of S contained in S.



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(4) $\{\vec{x} \in \mathbb{R}^n \mid 0<\|\vec{x}\|<1\} \subset \mathbb{R}^n$

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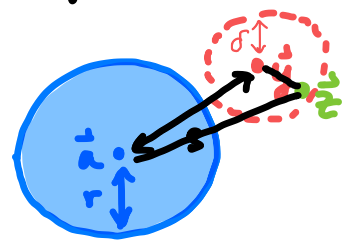
What do their closures and interiors look like?

In the HW, you'll show that a "closed rectangle" $[a_1,b_1]\times \dots \times [a_n,b_n] \subset \mathbb{R}^n$ is indeed closed. To give an idea of what sorts of proofs are involved, I will now show carefully that a "closed ball" is closed:

$\bar{B}(\vec{a},r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}-\vec{a}\| \le r\} \subset \mathbb{R}^n$ is a closed subset of $\mathbb{R}^n$

Proof: We have to show that the complement $\bar{B}^c := \mathbb{R}^n \setminus \bar{B}$ is open.



If $\vec{y} \in \bar{B}^c$, then $s := \|\vec{y}-\vec{a}\| > r$. Let $\delta := s-r$.

By the triangle inequality, for each $z \in B(\vec{y},\delta)$

$$\|\vec{z}-\vec{a}\| + \|\vec{y}-\vec{z}\| \ge \|\vec{y}-\vec{a}\|$$

Exercise Let $\mathcal{S} \subset \mathbb{R}^n$ be a set.

(1) Is it true that interior points of $\overline{\mathcal{S}}$ are points of $\mathcal{S}$?

(2) Is it true that the boundary of $\partial\mathcal{S}$ is $\partial\mathcal{S}$ itself?

$\mathbb{Q} \subset \mathbb{R}$

$\overline{B}(\vec{a}, r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| \le r\} \subset \mathbb{R}^n$ is a closed subset of $\mathbb{R}^n$

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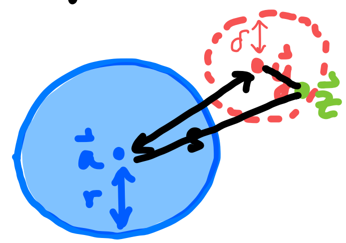


If $\vec{y} \in \overline{B}^c$, then $s := \|\vec{y} - \vec{a}\| > r$.

Let $\delta := s - r$.

By the triangle inequality, for each $z \in B(\vec{y}, \delta)$

$$\|\vec{z} - \vec{a}\| + \|\vec{y} - \vec{z}\| \ge \|\vec{y} - \vec{a}\| \implies$$

$$\|\vec{z} - \vec{a}\| \ge \|\vec{y} - \vec{a}\| - \|\vec{y} - \vec{z}\| > s - \delta = r$$

$\Rightarrow B(\vec{y}, \delta) \subset \overline{B}^c$. So $\overline{B}^c$ contains an open ball about each of its points, hence is open. $\square$

§2. Limits of sequences

A **sequence** is a function from

$$\mathbb{N} = \{1, 2, 3, \ldots\} \longrightarrow \mathbb{R}^n$$

$$k \longmapsto \vec{x}_k$$

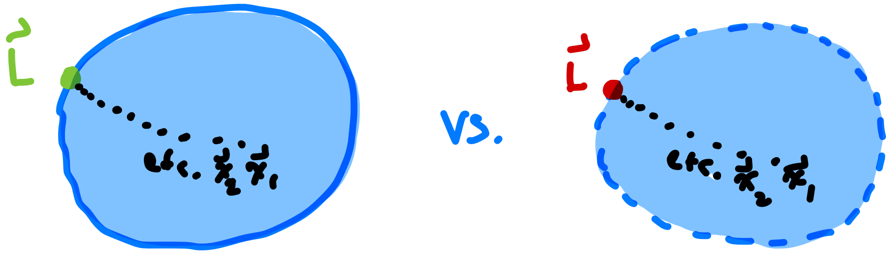
We say "$\{\vec{x}_k\}$ converges to $\vec{L}$" and write $\lim_{k\to\infty} \vec{x}_k = \vec{L}$ or $\vec{x}_k \to \vec{L}$

if

$$\boxed{\begin{array}{l} \forall \epsilon > 0 \ \exists K \in \mathbb{N} \text{ such that} \\ \quad k \geq K \implies \vec{x}_k \in B(\vec{L}, \epsilon) \end{array}}$$

If $\{\vec{x}_k\}$ converges to some $\vec{L}$, we say it **converges**; otherwise, it **diverges**.

Theorem: $S \subset \mathbb{R}^n$ is closed $\iff$ every convergent sequence of points in S converges to a point in S.



Proof: ($\Longrightarrow$) Suppose S is closed, and $\{\vec{x}_k\} \subset S$ is a sequence converging to $\vec{L} \notin S$. Since S^c is open and contains $\vec{L}$, it contains a ball $B(\vec{L}, \epsilon)$. Since $\vec{x}_k \to \vec{L}$, $\vec{x}_k \in B(\vec{L}, \epsilon) \subset S^c$ for $k \geq$ some K. But this contradicts $\vec{x}_k \in S$; and so $\vec{L} \in S$.

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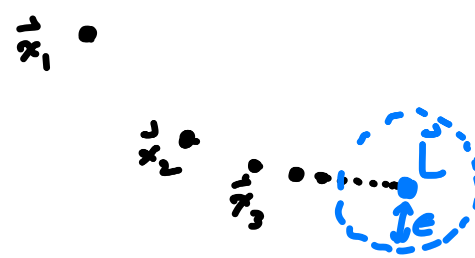

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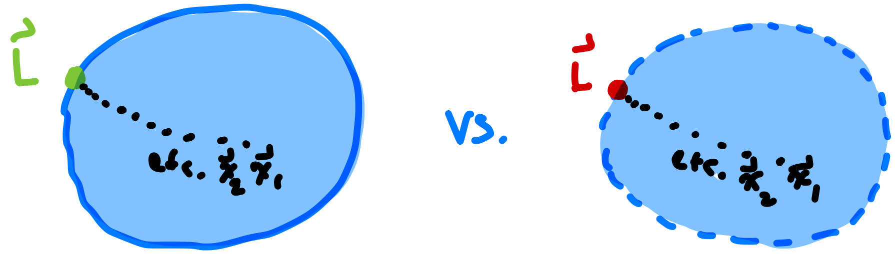

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($\Longleftarrow$) Suppose every convergent sequence in S has limit in S. If S is not closed, then S^c is not open — i.e. S^c contains a point $\vec{b}$ but no open ball about $\vec{b}$. So $B(\vec{b}, \frac{1}{k}) \cap S$ is nonempty ($\forall k$), and we take $\vec{x}_k$ in it. This gives a sequence $\vec{x}_k \to \vec{b}$. By assumption, since $\{\vec{x}_k\} \subset S$, we have $\vec{b} \in S$. This contradicts $\vec{b} \in S^c$, and so S must have been closed after all. $\square$

If $\mathcal{S}$ is not closed, then $\mathcal{S}^c$ is not open — i.e. $\mathcal{S}^c$ contains a point $\vec{b}$ but no open ball about $\vec{b}$. So $B(\vec{b}, \frac{1}{k}) \cap \mathcal{S}$ is nonempty $(\forall k)$, and we take $\vec{x}_k$ in it. This gives a sequence $\vec{x}_k \to \vec{b}$. By assumption, since $\{\vec{x}_k\} \subset \mathcal{S}$, we have $\vec{b} \in \mathcal{S}$. This contradicts $\vec{b} \in \mathcal{S}^c$, and so $\mathcal{S}$ must have been closed after all. $\square$

Quick Examples:

① $x_k = \dfrac{2^k}{\sqrt{2^{2k}+1}} \to 1$ Given $\epsilon > 0$, take $k > K := \left\lceil \frac{1}{2} \log_2\left((1-\epsilon)^{-2} - 1\right) \right\rceil$ to get $|x_k - 1| < \epsilon$.

easier: $\displaystyle\lim_{k\to\infty} x_k = \lim_{k\to\infty} \frac{1}{\sqrt{1+2^{-2k}}}$

$$= \frac{1}{\sqrt{1+\underbrace{\lim_{k\to\infty} 2^{-2k}}_{0}}} = 1$$

will be able to do this after Thursday

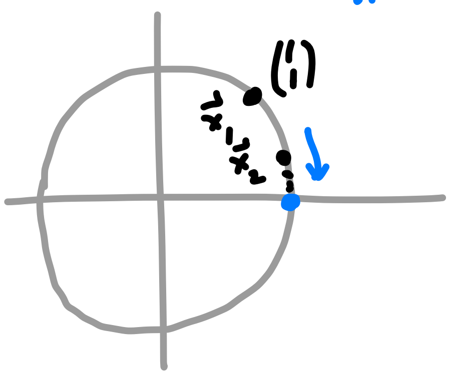
② $x_k = (-1)^k$ diverges

③ $A = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix}1\\1\end{pmatrix}$

⇝ sequence $\vec{x}_k := \dfrac{A^k \vec{x}_0}{\|A^k \vec{x}_0\|} \in \mathbb{R}^2$.

Here $\vec{x}_k = \dfrac{\begin{pmatrix}2^k & \\ & 1^k\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}{\| \quad " \quad \|} = \dfrac{1}{\sqrt{2^{2k}+1}}\begin{pmatrix}2^k\\1\end{pmatrix}$

$$= \begin{pmatrix} \dfrac{2^k}{\sqrt{2^{2k}+1}} \\[2ex] \dfrac{1}{\sqrt{2^{2k}+1}}\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}.$$



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② $x_k = (-1)^k$ diverges. Pick any $a \in \mathbb{R}$, $\epsilon < 1$. Need $K \in \mathbb{N}$ s.t. $k > K \Rightarrow$

$2 = |x_{k+1} - x_k| \le |x_{k+1} - a| + |x_k - a| < 2\epsilon < 2$ ✗

③ $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\rightsquigarrow$ sequence $\vec{x}_k := \dfrac{A^k \vec{x}_0}{\|A^k \vec{x}_0\|} \in \mathbb{R}^2$.

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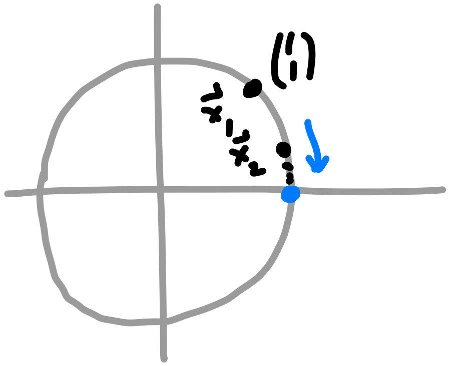

$= \begin{pmatrix} \dfrac{2^k}{\sqrt{2^{2k}+1}} \\ \dfrac{1}{\sqrt{2^{2k}+1}} \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Here I took limits componentwise:

$\begin{pmatrix} x_k \\ y_k \end{pmatrix} \to \begin{pmatrix} a \\ b \end{pmatrix} \iff x_k \to a$ & $y_k \to b$.

This will be justified next time.

§3. Limits of functions

We are interested in functions from a subset (usually open) of $\mathbb{R}^n$ to $\mathbb{R}^m$,

$\vec{F} : \mathcal{S} \longrightarrow \mathbb{R}^m$.

Two special cases:

- [parametric curve] $n = 1$

  $\vec{f} : I \longrightarrow \mathbb{R}^m$

  $\mathbb{R} \supset$ interval

Drawing a parametric curve or graphing a multivariable function are good ways to try to understand them. For instance, for $f : \mathbb{R}^2 \to \mathbb{R}$ the graph is the set of solutions to $f(x,y)=z$ in $\mathbb{R}^3$. The <u>level curves</u> $f(x,y)=k$ in $\mathbb{R}^2$ can help to visualize the graph.

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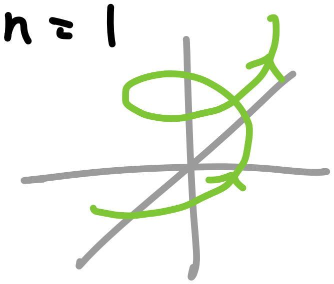
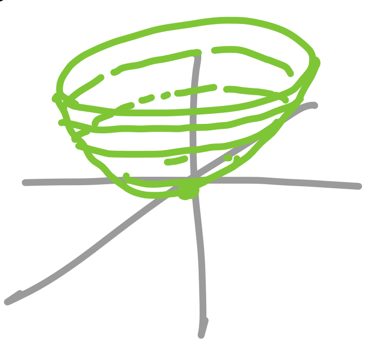


- [multivariable function] $m=1$

$$f : S \longrightarrow \mathbb{R}$$

$S \subset \mathbb{R}^n$



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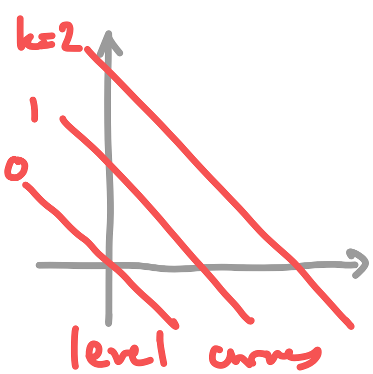
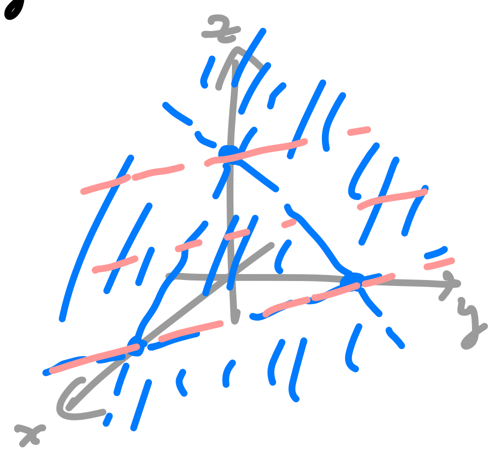
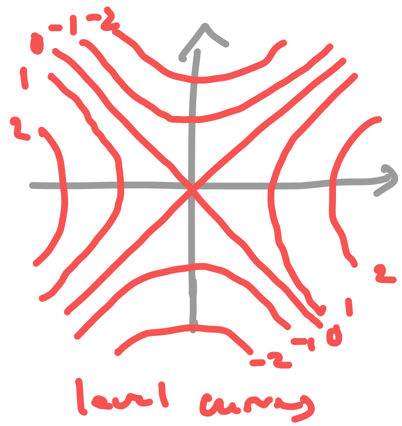
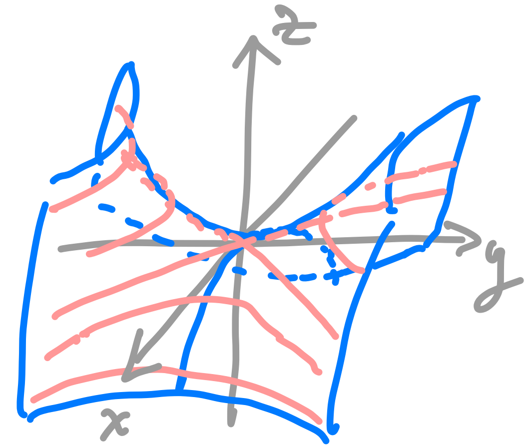
- $f(x,y) = 1 - x - y$





- $f(x,y) = y^2 - x^2$





e.g. for $n=2$, $m=1$ this reads:

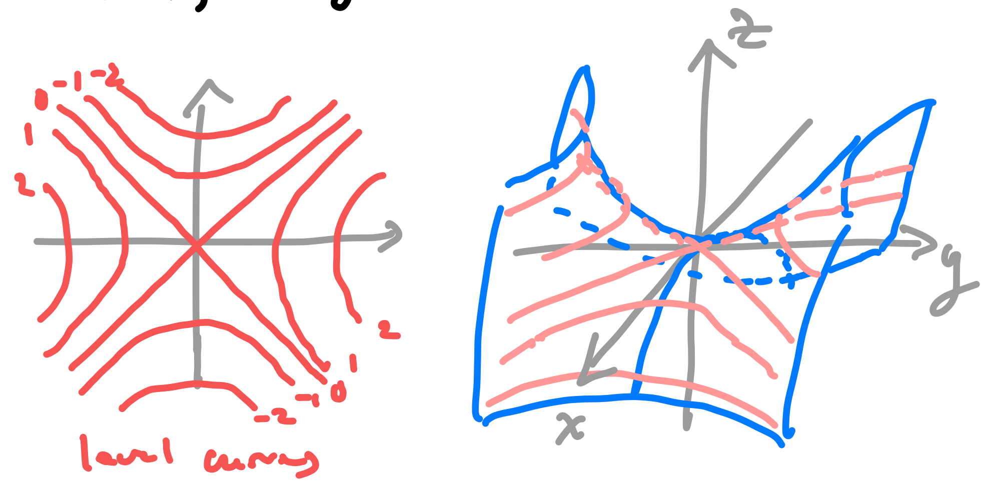
$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff$$

$\forall \epsilon > 0\ \exists \delta > 0$ s.t.

$$\sqrt{(x-a)^2+(y-b)^2} < \delta \implies |f(x,y)-L| < \epsilon.$$

Note: $B^*(\vec{a},\delta) := \{\vec{x} \in \mathbb{R}^n \mid 0 < \|\vec{x}-\vec{a}\| < \delta\}$ is the punctured ball of radius δ about $\vec{a}$. It is $B(\vec{a},\delta)$ without the point $\vec{a}$. For limits, we don't care about the value $\vec{F}(\vec{a})$ of $\vec{F}$ at $\vec{a}$. (For continuity, we do.)

- $f(x,y) = y^2 - x^2$



Now consider a set $\mathcal{S} \subset \mathbb{R}^n$ and a (vector-valued) function

$$\vec{F} : \mathcal{S} \to \mathbb{R}^m.$$

Let $\vec{a} \in \overline{\mathcal{S}}$.

$\boxed{\text{Definition 1}}$ $\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x}) = \vec{L} \iff$

$\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\vec{x} \in B^*(\vec{a},\delta) \implies F(\vec{x}) \in B(\vec{L},\epsilon).$$

$(\cap\, \mathcal{S})$

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$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff \forall \varepsilon>0\ \exists \delta>0 \text{ s.t. } \sqrt{(x-a)^2+(y-b)^2}<\delta \Rightarrow |f(x,y)-L|<\varepsilon.$$

$\boxed{\text{Definition 2}}$ $\vec{F}$ is <u>continuous</u> at $\vec{a}$

if (i) $\vec{F}(\vec{a})$ exists (i.e. $\vec{a}\in\mathcal{S}$, not just $\bar{\mathcal{S}}$)

(ii) $\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x})$ exists

and

(iii) they are equal.

The standard limit laws hold: writing "lim" for $\lim_{\vec{x}\to\vec{a}}$,

- $\lim(\alpha\vec{F}+\beta\vec{G}) = \alpha\lim\vec{F} + \beta\lim\vec{G}$
- $\boxed{\lim \vec{F}\cdot\vec{G} = \lim\vec{F}\cdot\lim\vec{G}}$
- $\lim\|\vec{F}\| = \|\lim\vec{F}\|$
- $\lim \frac{\vec{F}}{g} = \frac{\lim\vec{F}}{\lim g}$ provided $\lim g$ exists.

- for compositions, $(\vec{F}\circ\vec{G})(\vec{x}) := \vec{F}(\vec{G}(\vec{x}))$ has limit $\vec{F}(\lim\vec{G}(\vec{x}))$ if $\vec{F}$ is continuous at $\lim\vec{G}(\vec{x})$.

(I will prove a version of this next time.)

- To see how the dot product case works: if $\lim\vec{F}=\vec{L}$, $\lim\vec{G}=\vec{K}$,

$$\vec{F}\cdot\vec{G}-\vec{L}\cdot\vec{K} = (\vec{F}-\vec{L})\cdot(\vec{G}-\vec{K}) + \vec{L}\cdot(\vec{G}-\vec{K}) + \vec{K}\cdot(\vec{F}-\vec{L}).$$

Taking $\|\cdots\|$ on both sides and applying the triangle inequality, since $\|\vec{F}-\vec{L}\|$ and $\|\vec{G}-\vec{K}\| \to 0$, so does the left-hand side.

- hence so are its components $\vec{e}_k \cdot \vec{I}(\vec{x}) = x_k$ ...
- and polynomials $P(x_1, \dots, x_n): \mathbb{R}^n \to \mathbb{R}$, rational functions $\dfrac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)}$ ...
- and functions like $\sin(x^2 y)$ or $\log(x^2+y^2)$ or $|x-y|e^{x+y}$, etc. ...
- and vector-valued functions having these as components. (will explain next time)

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Taking $\|\cdots\|$ on both sides and applying the triangle inequality, since $\|\vec{F} - \vec{L}\|$ and $\|\vec{G} - \vec{K}\| \to 0$, so does the left-hand side.

The main point is that this gives us lots of continuous functions:

- the identity function $\vec{I}(\vec{x}) := \vec{x}$ is continuous

- hence so are its components $\vec{e}_k \cdot \vec{I}(\vec{x}) = x_k$ ...
- and polynomials $P(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$, rational functions $\dfrac{P(x_1, \ldots, x_n)}{Q(x_1, \ldots, x_n)}$ ...
- and functions like $\sin(x^2 y)$ or $\log(x^2+y^2)$ or $|x-y| e^{x+y}$, etc. ...
- and vector-valued functions having these as components. (will explain next time)

For scalar-valued functions, the definition of $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ implies that if you restrict the function to any path approaching $\vec{a}$, you approach the same L (independent of path).

For functions of 1 variable, this is just the statement that $x \to a^-$ and $x \to a^+$ need to yield the same limit. But in (say) 2 variables, paths could be line segments, parabolic segments, spirals, etc.

Since you can't "check all paths", this is mainly useful when 2 paths give different limits, allowing you to conclude that the limit DNE.

Actually it's interesting to write the first one in polar coordinates:

$$\frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} = \cos^2\theta - \sin^2\theta .$$

One sees in this way that the limit on a straight line path is determined by the angle of approach and can be any number between -1 & 1.

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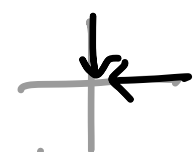
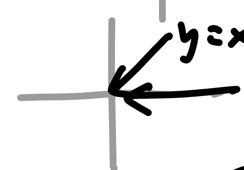
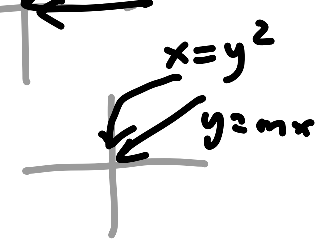
Example: To show $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE for

- $\dfrac{x^2-y^2}{x^2+y^2}$ try the paths 
- $\dfrac{xy}{x^2+y^2}$ " " 

- $\dfrac{xy^2}{x^2+y^4}$ " " 

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Since $r \to 0^+$ is equivalent to $(x,y) \to (0,0)$, this suggests a way to prove that such limits do exist:

Example "$\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} =$

$$= \lim_{r\to 0^+} \frac{3r^3\cos^2\theta \sin\theta}{r^2} = \lim_{r\to 0^+} 3r\cos^2\theta \sin\theta$$

$= 0$" is a bit heuristic, but essentially correct.

The completely correct way to do this is to say that on $B(\vec{0},\delta)$ (i.e. $r<\delta$)

$$\left|\frac{3x^2y}{x^2+y^2}\right| < 3\delta\,|\cos^2\theta \sin\theta| \le 3\delta$$

(so taking $\delta = \varepsilon/3$ we're done).

HW #1 due tomorrow via Canvas upload by noon.

OH tonight 8–9 PM.