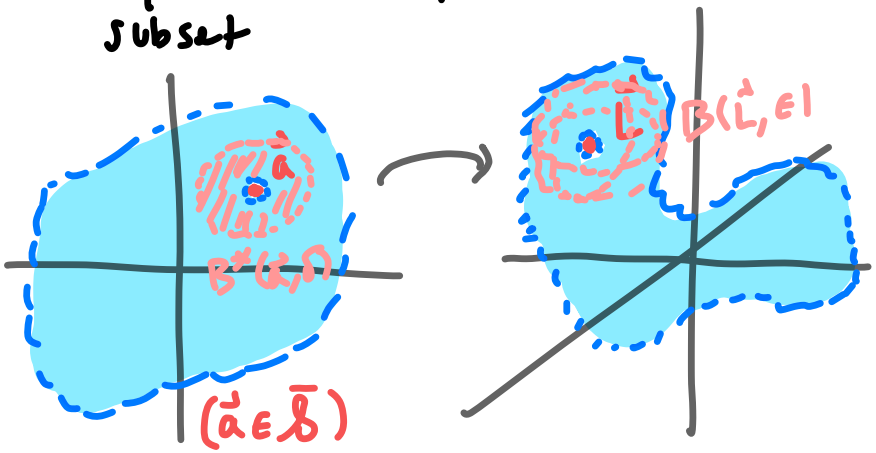
CONTINUITY AND PARTIALS

4

§ 1. Review of Continuity

$$\mathbb{R}^n \supset \mathcal{S} \xrightarrow[\vec{F}]{} \mathbb{R}^m$$

open subset

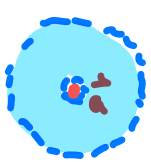


$$\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x}) = \vec{L} \overset{\text{def.}}{\iff}$$

$\forall \epsilon > 0 \ \exists \delta > 0$ such that

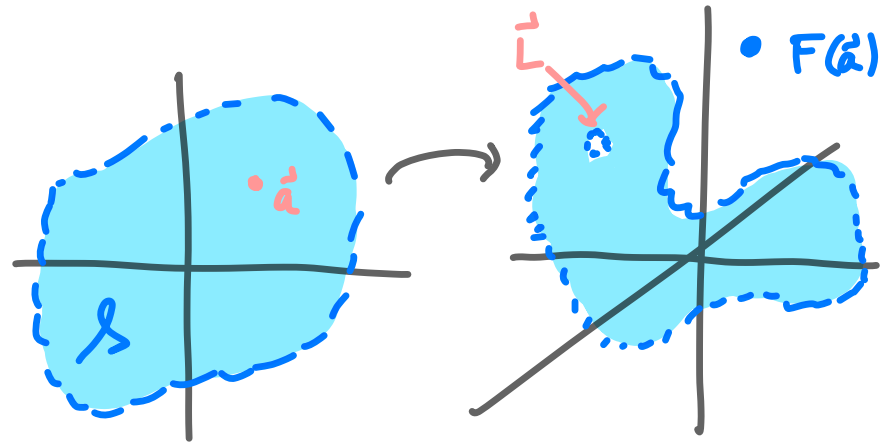
$$\left[\vec{x} \in B^*(\vec{a},\delta) \cap \mathcal{S} \Rightarrow F(\vec{x}) \in B(\vec{L},\epsilon)\right]$$

punctured ball $B(\vec{a},\delta)\setminus\{\vec{a}\}$:



$0 < \|\vec{x}-\vec{a}\| < \delta$

This picture also has $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \vec{L}$:



but is an example of what continuity rules out:

$\vec{F}$ is <u>continuous at</u> $\vec{a} \in \mathcal{S}$ ($\vec{a}$ must be in the domain)

$$\overset{\text{def.}}{\iff} \lim_{\vec{x}\to\vec{a}} F(\vec{x}) = F(\vec{a})$$

↳ exists and

$$\iff \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that}$$

$$\left[\vec{x} \in B(\vec{a},\delta) \Longrightarrow F(\vec{x}) \in B(\vec{F}(\vec{a}),\epsilon)\right]$$

$(\subset \mathcal{S})$

Example: $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

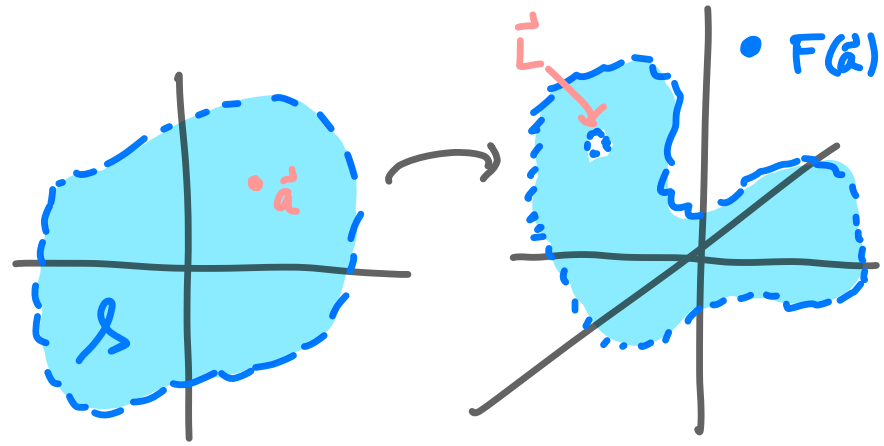
is not continuous at $(0,0)$ because the limits

along $(x,y) = (t, mt)$: $\dfrac{m^2 t^3}{t^2(1+m^4 t^2)} \to 0$ $(t \to 0)$

along $(x,y) = (t^2, t)$: $\dfrac{t^4}{2t^4} \to \dfrac{1}{2}$

disagree.

This picture also has $\lim_{\vec{x} \to \vec{a}} F(\vec{x}) = \vec{L}$:



but is an example of what continuity rules out:

$\vec{F}$ is <u>continuous at $\vec{a} \in S$</u> ($\vec{a}$ must be in the domain)

def. $\iff \lim_{\vec{x} \to \vec{a}} F(\vec{x}) = F(\vec{a})$

↳ exists and

$\iff \forall \epsilon > 0 \ \exists \delta > 0$ such that

$$\left[\vec{x} \in B(\vec{a}, \delta) \implies F(\vec{x}) \in B(F(\vec{a}), \epsilon) \right]$$

$(\subset S)$

$\vec{F}$ is <u>continuous</u> $\iff$ this holds at every $\vec{a} \in S$.

Example: $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

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disagree.

Going back to limits of

$$\mathbb{R}^n \supset S \xrightarrow[\vec{F} = (F_1, \dots, F_m)]{} \mathbb{R}^m$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} F_1(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{pmatrix}$$

I want to point out that

$$\lim_{\vec{x} \to \vec{a}} \vec{F}(\vec{x}) = \vec{L} \iff \lim_{\vec{x} \to \vec{a}} F_j(\vec{x}) = L_j$$

$$\vec{L} = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix}$$

for $j = 1, \dots, m$.

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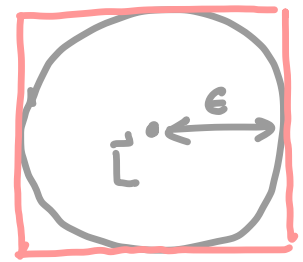
Proof: Let $\epsilon > 0$ be given.

($\Rightarrow$) Suppose $\lim_{\vec{x} \to \vec{a}} \vec{F}(\vec{x}) = \vec{L}$. Then for some $\delta > 0$,

$\vec{x} \in B(\vec{a}, \delta) \cap S \Rightarrow$

$\|\vec{F}(\vec{x}) - \vec{L}\| < \epsilon \Rightarrow$

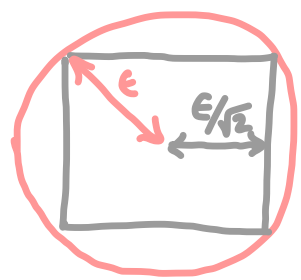
$|F_j(\vec{x}) - L_j| < \epsilon$.



($\Leftarrow$) Suppose $\lim_{\vec{x} \to \vec{a}} F_j(\vec{x}) = L_j \ (\forall j)$.

Then $\exists\, \delta > 0$ s.t. $\vec{x} \in B(\vec{a}, \delta) \cap S$

$$\Downarrow$$



$$|F_j(\vec{x}) - L_j| < \frac{\epsilon}{\sqrt{m}} \quad (\forall j)$$

$$\Downarrow$$

$$\|\vec{F}(\vec{x}) - \vec{L}\| = \sqrt{\sum_{j=1}^{m} |F_j(\vec{x}) - L_j|^2}$$

$$< \sqrt{\sum_{j=1}^{m} \epsilon^2/m} = \sqrt{\epsilon^2} = \epsilon.$$

$\square$

Going back to limits of

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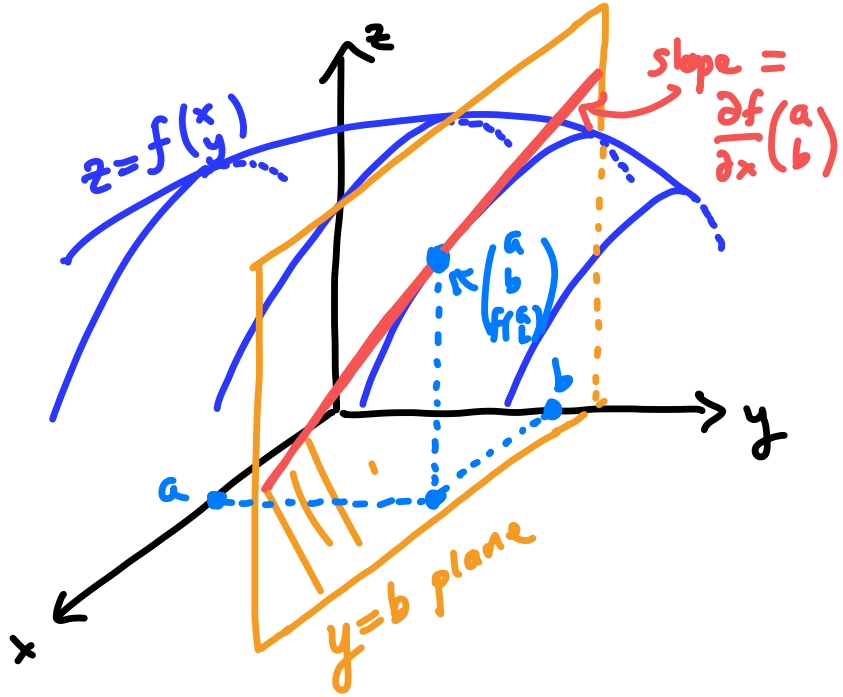
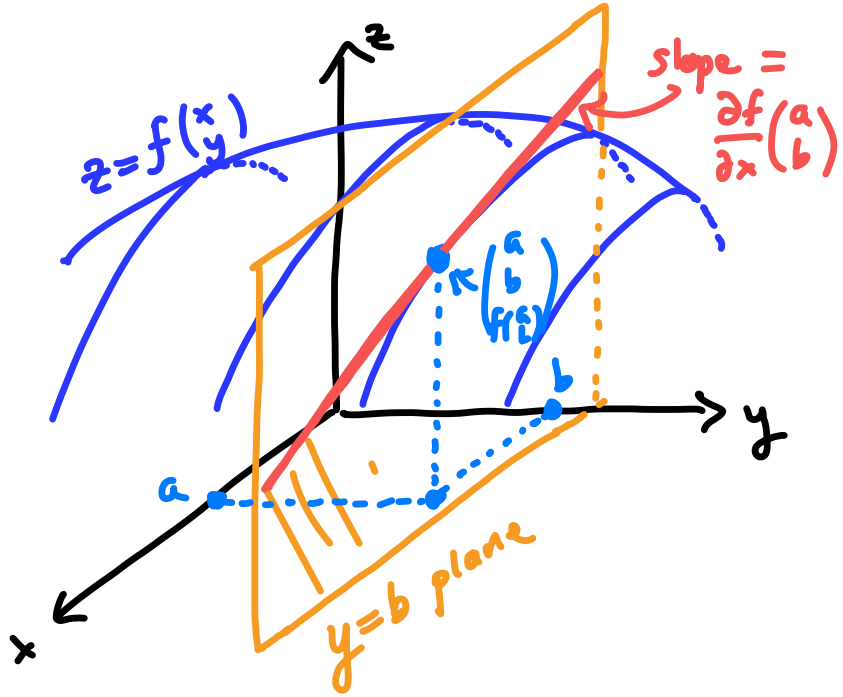
This reduces computing limits or checking continuity of vector-valued functions to doing it on each (scalar-valued) component F_j.



§2. Partial derivatives

Given a (scalar-valued) function $f : S \to \mathbb{R}$ defined on an open set $S \subset \mathbb{R}^n$, the idea of partial derivatives is to hold all variables but one — say, x_k — constant, and differentiate with respect to x_k to get "$\partial f / \partial x_k$". If $n=2$, we often write $\binom{x}{y}$ instead of $\vec{x} = \binom{x_1}{x_2}$, and the geometric idea is to slice the graph $z = f\binom{x}{y}$ by the plane $y=b$, then compute the slope of $z = f\binom{x}{b}$ at $x=a$:

$$\frac{\partial f}{\partial x}\binom{a}{b} := \lim_{h \to 0} \frac{f\binom{a+h}{b} - f\binom{a}{b}}{h}.$$



More generally,

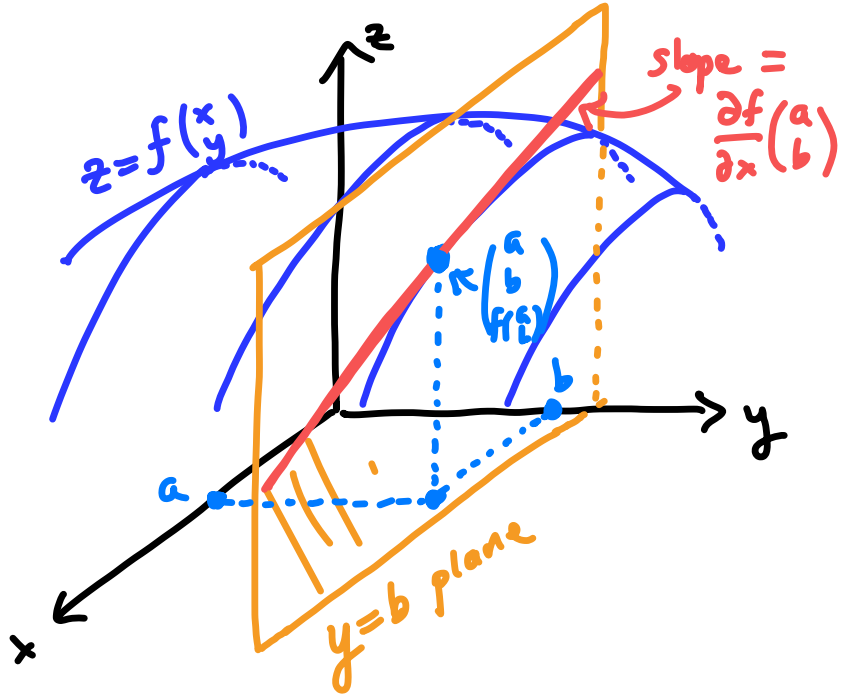
$$\frac{\partial f}{\partial x} := \lim_{h \to 0} \frac{f\binom{x+h}{y} - f\binom{x}{y}}{h} \quad \& \quad \frac{\partial f}{\partial y} := \lim_{k \to 0} \frac{f\binom{x}{y+k} - f\binom{x}{y}}{k}.$$

$\boxed{\text{Exercise}}$ Compute the partials of

$f\binom{x}{y} := \sin\left(\frac{x}{1+y}\right)$. What can you say about partials of

$$f\binom{x}{y} := \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

at $\vec{0}$?



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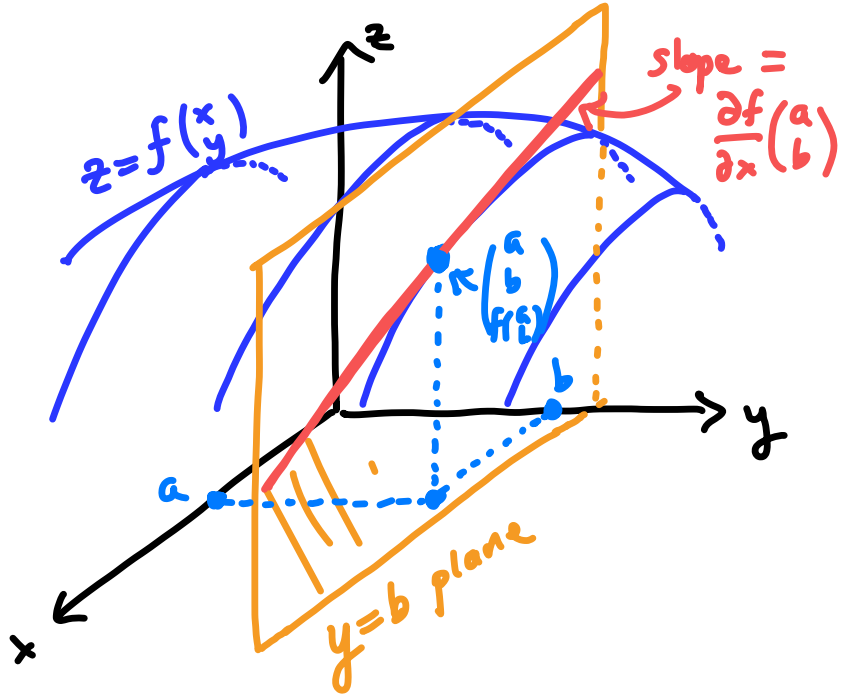
① $\frac{\partial f}{\partial x} = \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right)$ and

$$\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial y} \frac{x}{1+y}\right) \cos\left(\frac{x}{1+y}\right) = \frac{-x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right)$$



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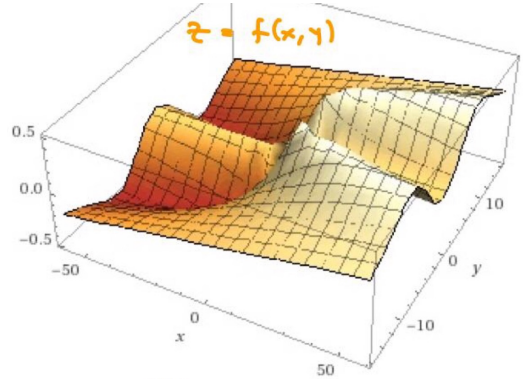
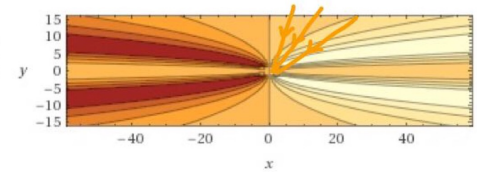


The graph



Contour plot

(level curves)

— I have drawn a few of the linear paths into (0,0) that gave limits = 0.

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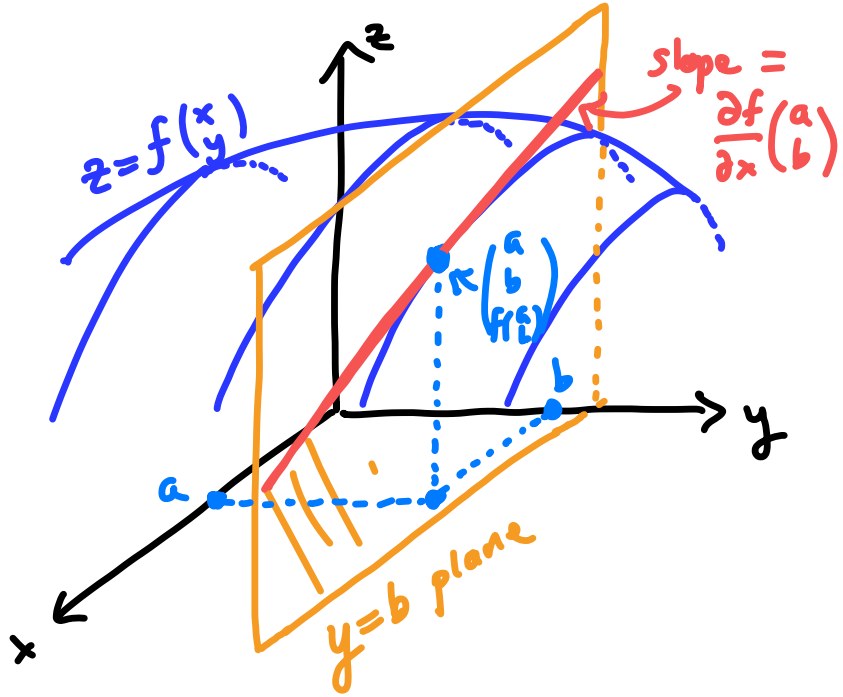
While this one is discontinuous at $\vec{0}$, the partials exist there: restricting to $y=0$ and differentiating with respect to x gives

$$\frac{\partial f}{\partial x}\binom{0}{0} = \frac{d}{dx} f\binom{x}{0}\Big|_{x=0} = \frac{d}{dx} 0\Big|_{x=0} = 0;$$

while restricting to $x=0$ and differentiating w.r.t. y gives

$$\frac{\partial f}{\partial y}\binom{0}{0} = \frac{d}{dy} f\binom{0}{y}\Big|_{y=0} = \frac{d}{dy} 0\Big|_{y=0} = 0.$$



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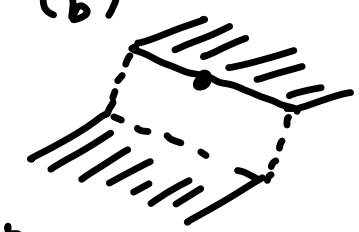
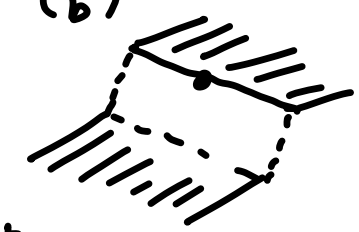
There are essentially 3 cases to consider:

- partials exist & are continuous at $\binom{a}{b}$
  $\Rightarrow z=f\binom{x}{y}$ is well-approximated by a "tangent plane" at $\begin{pmatrix} a \\ b \\ f\binom{a}{b} \end{pmatrix}$.

- partials don't exist at $\binom{a}{b}$ — something like or 

- partials exist but aren't continuous at $\binom{a}{b}$ — the weird in-between case we found ourselves in in ②.

We won't consider $f\binom{x}{y}$ "differentiable" at $\binom{a}{b}$ in the last two cases.

- partials don't exist at $\binom{a}{b}$ — something like or 
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For $\mathbb{R}^n \supset S \xrightarrow{\vec{F}} \mathbb{R}^m$, partials are defined by

$$\frac{\partial \vec{F}}{\partial x_j}(\vec{a}) := \lim_{t \to 0} \frac{\vec{F}(\vec{a}+t\hat{e}_j) - \vec{F}(\vec{a})}{t}.$$

A slight generalization is the directional derivative in the direction $\vec{v}$:

$$D_{\vec{v}}\vec{F}(\vec{a}) := \lim_{t \to 0} \frac{\vec{F}(\vec{a}+t\vec{v}) - F(\vec{a})}{t}.$$

- $D_{\vec{0}}\vec{F}(\vec{a}) = \vec{0}$
- $D_{c\vec{v}}\vec{F}(\vec{a}) = c\, D_{\vec{v}}\vec{F}(\vec{a})$

  since replacing t in $\dfrac{\vec{F}(\vec{a}+tc\vec{v}) - F(\vec{a})}{t}$

  by $s = ct$ gives $c\,\dfrac{\vec{F}(\vec{a}+s\vec{v}) - F(\vec{a})}{s}$
- For scalar-valued functions, we can think of it as giving slope in the $\vec{v}$-direction if $\|\vec{v}\| = 1$
- partial derivatives are the special case $\vec{v} = \hat{e}_j$.

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  $$\frac{\partial \vec{F}}{\partial x_j} = \begin{pmatrix} \partial F_1/\partial x_j \\ \vdots \\ \partial F_m/\partial x_j \end{pmatrix}$$ can be computed

  component-wise, since we already showed this for limits.

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Example: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0, & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$

To compute the directional derivative in the direction $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ at $\vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, write $f(t\vec{v}) = f\begin{pmatrix} \alpha t \\ \beta t \end{pmatrix} = \dfrac{\alpha\beta^2 t}{\alpha^2 + \beta^4 t^2}$. So

$$D_{\vec{v}} f(\vec{0}) = \lim_{t\to 0} \frac{\left(\dfrac{\alpha\beta^2 \cancel{t}}{\alpha^2+\beta^4 t}\right)}{\cancel{t}} = \frac{\alpha\beta^2}{\alpha^2} = \frac{\beta^2}{\alpha}$$

if $\alpha \neq 0$ (and 0 if $\alpha = 0$).

Conclude that all the directional derivatives exist at $\vec{0}$!

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Conclude that all the directional derivatives exist at $\vec{0}$!

If we limit ourselves to unit vectors $\vec{v} = \begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$, we get "directional slopes"

$$\begin{cases} \dfrac{\sin^2\theta}{\cos\theta} & \text{if } \cos\theta \neq 0 \\ 0 & \text{if } \cos\theta = 0. \end{cases}$$

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§3. Continuity & open sets

Theorem: $\vec{F} : \mathcal{S} \to \mathbb{R}^m$ is continuous ($\mathcal{S} \subset \mathbb{R}^n$)

$\iff \vec{F}^{-1}(V) \subset \mathbb{R}^n$ is open for every open $V \subset \mathbb{R}^m$

Proof: Recall that continuity means that for each $\vec{a} \in \mathcal{S}$ and $\epsilon > 0$, we have $\vec{F}(B(\vec{a},\delta)) \subset B(\vec{F}(\vec{a}), \epsilon)$ for some $\delta > 0$.

($\Leftarrow$): Given $\vec{a}$ & ϵ, take $V := B(\vec{F}(\vec{a}), \epsilon)$. Since $\vec{F}^{-1}(V)$ is open & contains $\vec{a}$, it contains a ball $B(\vec{a},\delta)$ about $\vec{a}$. Hence $\vec{F}(B(\vec{a},\delta)) \subset V$.

($\Rightarrow$): Given $V \subset \mathbb{R}^m$ open and $\vec{a} \in \vec{F}^{-1}(V)$, we need to fit a ball about $\vec{a}$ inside $\vec{F}^{-1}(V)$. Since V is open, we do have a ball $B(\vec{F}(\vec{a}), \epsilon) \subset V$.

By assumption, $\vec{F}$ is continuous at $\vec{a}$; so $\exists\, \delta > 0$ with

$$\vec{F}(B(\vec{a},\delta)) \subset B(\vec{F}(\vec{a}), \epsilon) \subset V,$$

hence $B(\vec{a},\delta) \subset \vec{F}^{-1}(V)$. $\square$

This makes it really easy to show

Corollary: Given continuous functions

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\vec{G}"] & \mathcal{T} \arrow[r, "\vec{F}"] & \mathbb{R}^m \\ \mathbb{R}^p \arrow[u, phantom, "\cap"] & \mathbb{R}^n \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

— open —

the composition $\vec{F} \circ \vec{G}$ is continuous.

Proposition: $\vec{F}: \mathcal{S} \to \mathbb{R}^m$ is continuous at $\vec{a} \in \mathcal{S}$

$\iff$ every sequence $\{\vec{x}_k\} \subset \mathcal{S}$ with $\vec{x}_k \to \vec{a}$ has $F(\vec{x}_k) \to F(\vec{a})$.

By assumption, $\vec{F}$ is continuous at $\vec{a}$; so $\exists\, \delta > 0$ with

$$\vec{F}(B(\vec{a},\delta)) \subset B(\vec{F}(\vec{a}), \epsilon) \subset V,$$

hence $B(\vec{a},\delta) \subset \vec{F}^{-1}(V)$. $\square$

This makes it really easy to show

Corollary: Given continuous functions

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\vec{G}"] & \mathcal{J} \arrow[r, "\vec{F}"] & \mathbb{R}^m \\ \mathbb{R}^p \arrow[u, phantom, "\cap"] \arrow[r, phantom, "\text{— open —}"] & \mathbb{R}^n \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

the composition $\vec{F} \circ \vec{G}$ is continuous.

Proof: $V \subset \mathbb{R}^m$ open $\underset{\vec{F}\text{ cts.}}{\Longrightarrow}$

$\vec{F}^{-1}(V)$ open $\underset{\vec{G}\text{ cts}}{\Longrightarrow}$

$\vec{G}^{-1}(\vec{F}^{-1}(V))$ open

$\parallel$

$(\vec{F} \circ \vec{G})^{-1}(V)$ open! $\square$

Proposition: $\vec{F}: S \to \mathbb{R}^m$ is continuous at $\vec{a} \in S$

$\iff$ every sequence $\{\vec{x}_k\} \subset S$ with $\vec{x}_k \to \vec{a}$ has $F(\vec{x}_k) \to F(\vec{a})$.

Proof: ($\Longrightarrow$) is clear: if $\vec{x}_k \to \vec{a}$, then $\exists K$ s.t. $k \geq K \Rightarrow \|\vec{x}_k - \vec{a}\| < \delta$

$$\underset{\vec{F} \text{ cts. at } \vec{a}}{\Longrightarrow} \|\vec{F}(\vec{x}_k) - \vec{F}(\vec{a})\| < \epsilon.$$

($\Longleftarrow$): Use the contrapositive [if the LHS fails, then the RHS fails]: Suppose $\vec{F}$ is not continuous at $\vec{a}$. Then for some $\epsilon_0 > 0$ $\nexists\, \delta$ satisfying the criterion — i.e. for every $\delta > 0$, there is an $\vec{x} \in B(\vec{a}, \delta)$ with $\vec{F}(\vec{x}) \notin B(\vec{F}(\vec{a}), \epsilon_0)$. Taking $\delta = \frac{1}{k}$, we get points $\vec{x}_k \in B(\vec{a}, \frac{1}{k})$ (thus $\vec{x}_k \to \vec{a}$) such that

$\|\vec{F}(\vec{x}_k) - \vec{F}(\vec{a})\| \geq \epsilon_0$, so

$\vec{F}(\vec{x}_k) \not\to \vec{F}(\vec{a})$. $\square$

Proposition: $\vec{F}: S \to \mathbb{R}^m$ is continuous at $\vec{a} \in S$

$\iff$ every sequence $\{\vec{x}_k\} \subset S$ with $\vec{x}_k \to \vec{a}$ has $F(\vec{x}_k) \to F(\vec{a})$.

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$\xRightarrow[\vec{F} \text{ cts. at } \vec{a}]{} \|\vec{F}(\vec{x}_k) - \vec{F}(\vec{a})\| < \epsilon$.

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$\|\vec{F}(\vec{x}_k) - \vec{F}(\vec{a})\| \geq \epsilon_0$, so

$\vec{F}(\vec{x}_k) \not\to \vec{F}(\vec{a})$. $\square$

Corollary: (Take $S = \mathbb{R}^n$.)

$\vec{F}$ continuous $\Rightarrow$ $\vec{F}^{-1}(\vec{c})$ is closed for each $\vec{c}$.

level sets

Proof: Given any convergent sequence $\{\vec{x}_k\} \subset \vec{F}^{-1}(\vec{c})$, we need to show its limit $\vec{a}$ lies in $\vec{F}^{-1}(\vec{c})$.

But $\vec{x}_k \to \vec{a}$ + continuity $\Rightarrow$ $\vec{F}(\vec{x}_k) \to \vec{F}(\vec{a})$, and every $\vec{F}(\vec{x}_k) = \vec{c}$. So $\vec{F}(\vec{a}) = \vec{c}$. $\square$

§4. Cauchy sequences (for HW problem §2.2 #14)

Given a sequence $\{\vec{x}_k\} \subset \mathbb{R}^n$, a subsequence is formed by taking $\vec{x}_{k_1}, \vec{x}_{k_2}, \ldots, \vec{x}_{k_j}, \ldots$ where $k_1 < k_2 < k_3 < \cdots$.

If $\vec{x}_k \to \vec{a}$, then $\vec{x}_{k_j} \to \vec{a}$ for any subsequence. But convergence of a subsequence does not imply convergence of the sequence (e.g. $(-1)^k$).

$\{\vec{x}_k\}$ is a Cauchy sequence if, for every $\epsilon > 0$, $\exists\, K \in \mathbb{N}$ such that $k, l > K \Rightarrow \|\vec{x}_k - \vec{x}_l\| < \epsilon$. [Any convergent sequence is Cauchy: use $\|\vec{x}_k - \vec{x}_l\| \le \|\vec{x}_k - \vec{a}\| + \|\vec{a} - \vec{x}_l\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.]

If a subsequence of a Cauchy sequence converges, then so does the sequence!

[use $\|\vec{x}_k - \vec{a}\| \le \|\vec{x}_k - \vec{x}_{k_j}\| + \|\vec{x}_{k_j} - \vec{a}\|$
$< \epsilon/2 + \epsilon/2 = \epsilon$]

Cauchy sequences are bounded — that is, they lie in some ball $B(R, \vec{0})$ about $\vec{0}$.

[take $\epsilon = 1$, and choose K so $l > K \Rightarrow$ $\|\vec{x}_l\| \le \|\vec{x}_l - \vec{x}_{K+1}\| + \|\vec{x}_{K+1}\| < 1 + \|\vec{x}_{K+1}\|$.
So every $\|\vec{x}_j\| \le \max(\|\vec{x}_1\|, \ldots, \|\vec{x}_K\|, 1 + \|\vec{x}_{K+1}\|)$.]

You will show in #14 that every Cauchy sequence in $\mathbb{R}^n$ converges. This needs one more idea, the monotone sequence theorem.

A subset $S \subset \mathbb{R}$ is bounded above if $\exists\, b \in \mathbb{R}$ such that $a \le b$ for all $a \in S$.

A least upper bound β for S is an upper bound which is $\le$ every upper bound.

Fact: If $S \subset \mathbb{R}$ is bounded above, it has a (unique!) least upper bound, $\sup(S)$.

Monotonic sequence theorem: If a_k is increasing and bounded above, then $\lim_{k\to\infty} a_k$ exists.

Proof: Set $L := \sup\{a_n\}$. Let $\epsilon > 0$. By definition, $L - \epsilon$ is not an U.B. That is, $\exists\, K \in \mathbb{N}$ s.t. $a_K > L - \epsilon$. Since a_k is increasing, $k \ge K \Rightarrow$

$$L \underset{\text{U.B.}}{\ge} a_k \ge a_K > L - \epsilon \Rightarrow$$

$$0 \ge a_k - L > -\epsilon \Rightarrow |a_k - L| < \epsilon.$$

Therefore $\lim_{k\to\infty} a_k = L$. $\square$

Corollary: If $I_1 \supset I_2 \supset I_3 \supset \cdots$ are closed intervals $[a_k, b_k]$, then $\bigcap_{k\ge 1} I_k$ is nonempty.

Proof: Notice that a_k is an increasing sequence, bounded above by any b_j. So by the Theorem, it converges to $L := \sup\{a_k\}$. We have $a_k \le L$ $(\forall k)$, and $L \le b_k$ $(\forall k)$ since L is the least U.B. Hence $L \in [a_k, b_k]$ for every k. $\square$

$\boxed{\text{Exercise}}$ Show that the sequence

$x_1 = 1, \quad x_k = 3 - \frac{1}{x_{k-1}}$ converges

(and compute its limit):

- show $1 \le x_{k-1} \le x_k \le 3$ by induction
- show $L = 3 - \frac{1}{L}$, and find L.

<u>Fact</u>: If $S \subset \mathbb{R}$ is bounded above, it has a (unique!) least upper bound, $\underline{\sup(S)}$.

<u>Monotonic sequence theorem</u>: If a_k is increasing and bounded above, then $\lim_{k\to\infty} a_k$ exists.

<u>Proof</u>: Set $L := \sup\{a_n\}$. Let $\epsilon > 0$. By definition, $L - \epsilon$ is not an U.B. That is, $\exists\, K \in \mathbb{N}$ s.t. $a_K > L - \epsilon$. Since a_k is increasing, $k \ge K \Rightarrow$

$$L \underset{\text{U.B.}}{\ge} a_k \ge a_K > L - \epsilon \quad \Rightarrow$$

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<u>Corollary</u>: If $I_1 \supset I_2 \supset I_3 \supset \cdots$ are closed intervals $[a_k, b_k]$, then $\bigcap_{k\ge 1} I_k$ is nonempty.

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