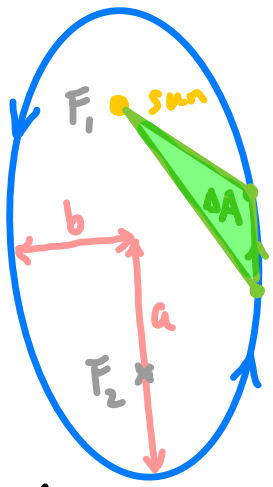
CURVES

6

In the 1st quarter of the 17th C., through his study of astronomical data collected by Brahe, Kepler was led to propose the following

LAWS OF PLANETARY MOTION:

① The motion is elliptical (not circular, as Copernicus had it), with the sun at one focus of the ellipse.



② Area is swept out at a constant rate (that is, $\frac{dA}{dt} = \text{const.}$)

③ The period of a revolution (= time to go once around the sun) is proportional to $a^{3/2}$, where a = length of semimajor axis

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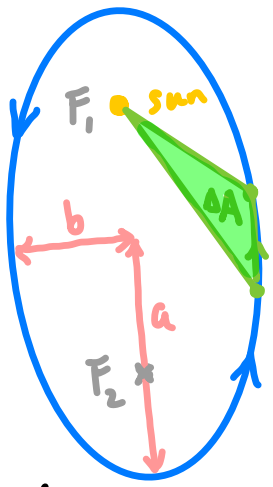

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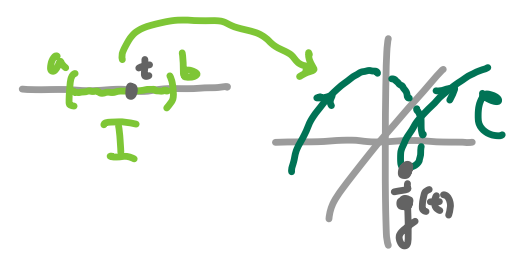
One of the great triumphs of "Calculus" was Newton's proof, nearly 70 years later, that Kepler's laws follow from Newton's 2nd law ($\vec{F}(t) = m\,\vec{a}(t)$, where $\vec{F}$ is the vector sum of all the forces acting on an object and $\vec{a}$ is its acceleration) and Law of Universal Gravitation.

To appreciate (among other things) this profound result, we turn to the Calculus of vector-valued functions

interval in $\mathbb{R}$

$$\vec{g} = (g_1, \dots, g_n) : I \longrightarrow \mathbb{R}^n$$
$$t \longmapsto (g_1(t), \dots, g_n(t))$$

parametrizing a curve $C := \vec{g}(I)$.



§1. Curvilinear motion

Assuming $\vec{g}$ is twice differentiable,

$\vec{g}(t)$ (or $\vec{r}(t)$) = position

$\vec{g}'(t)$ (or $\vec{v}(t)$) = velocity

$\|\vec{g}'(t)\|$ (or $v(t)$) = speed

$\vec{g}''(t)$ (or $\vec{a}(t)$) = acceleration.

One imagines a particle moving along C, whose instantaneous direction of motion at time t_0 is $\vec{g}'(t_0) = (g_1'(t_0), \dots, g_n'(t_0))$. If this is nonzero, the line parametrized by $\vec{l}(u) = \vec{g}(t_0) + u\vec{g}'(t_0)$ is called the tangent line to C at $\vec{g}(t_0)$, and $\hat{T}(t_0) := \dfrac{\vec{g}'(t_0)}{\|\vec{g}'(t_0)\|}$ the unit tangent vector. We say C is smooth at t_0 if $\lim_{t \to t_0} \hat{T}(t_0)$ exists, and smooth when it is smooth at each point (roughly: no corners or cusps).

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Question: What happens if our curve is on a sphere, i.e. if $\|\vec{g}(t)\|$ is constant?

$$\|\vec{g}(t)\| = \text{const.} \iff \vec{g}(t)\cdot\vec{g}(t) = \|\vec{g}(t)\|^2 = \text{const.}$$

$$\Big\downarrow \tfrac{d}{dt}$$

$$\iff 2\vec{g}'(t)\cdot\vec{g}(t) = 0$$

$$\iff \vec{g}'(t) \perp \vec{g}(t) \quad (\text{for all } t \in I).$$

EXAMPLE : $\boxed{\text{ELLIPTIC BILLIARDS}}$

Let $\vec{g}(t)$ trace out an ellipse C with foci $\vec{F}_1$ & $\vec{F}_2$. All we need to know is that $d_1 + d_2 = \|\vec{g}(t) - \vec{F}_1\| + \|\vec{g}(t) - \vec{F}_2\|$ is constant.

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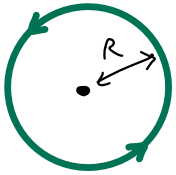

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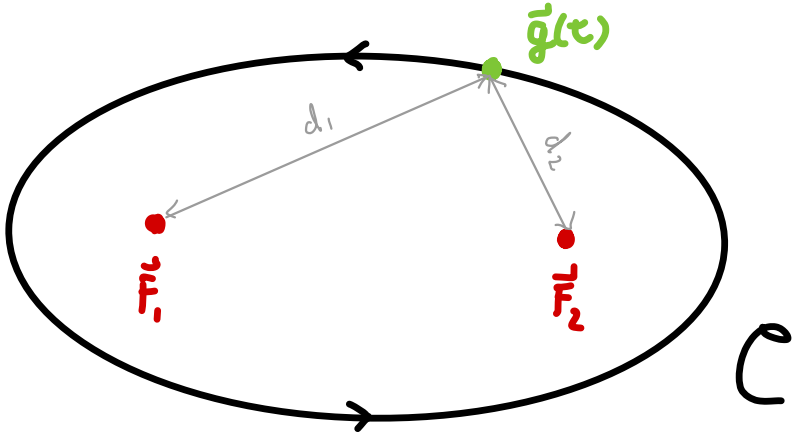

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Write $\vec{g}(t) - \vec{F}_i = d_i(t)\,\hat{u}_i(t)$, $\hat{u}_i$ = unit vector.

Then $d_i = (\vec{g} - \vec{F}_i)\cdot\hat{u}_i \implies$

$$d_i' = (\underbrace{\vec{g} - \vec{F}_i}_{\text{constant}})'\cdot\hat{u}_i + \underbrace{(\vec{g} - \vec{F}_i)}_{\text{parallel to } \hat{u}_i}\cdot\underbrace{\hat{u}_i'}_{\perp \text{ to } \hat{u}_i \text{ by } (*)} = \vec{g}'\cdot\hat{u}_i$$

so dot product $= 0$

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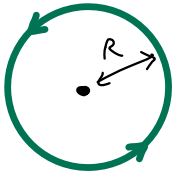

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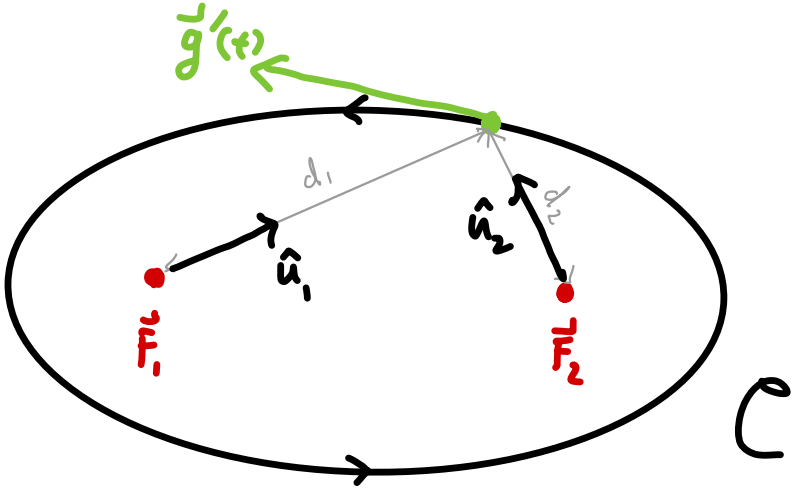

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while $d_1 + d_2 = \text{const.} \implies$

$$0 = d_1' + d_2' = \vec{g}'\cdot\hat{u}_1 + \vec{g}'\cdot\hat{u}_2 \implies$$

divide by $\|\vec{g}'\|$

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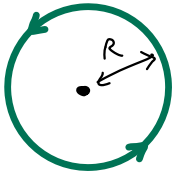

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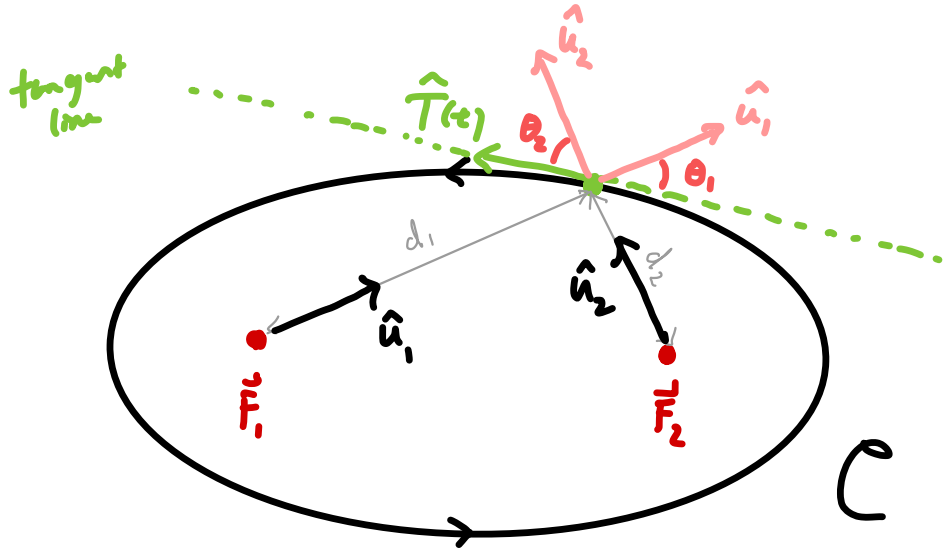

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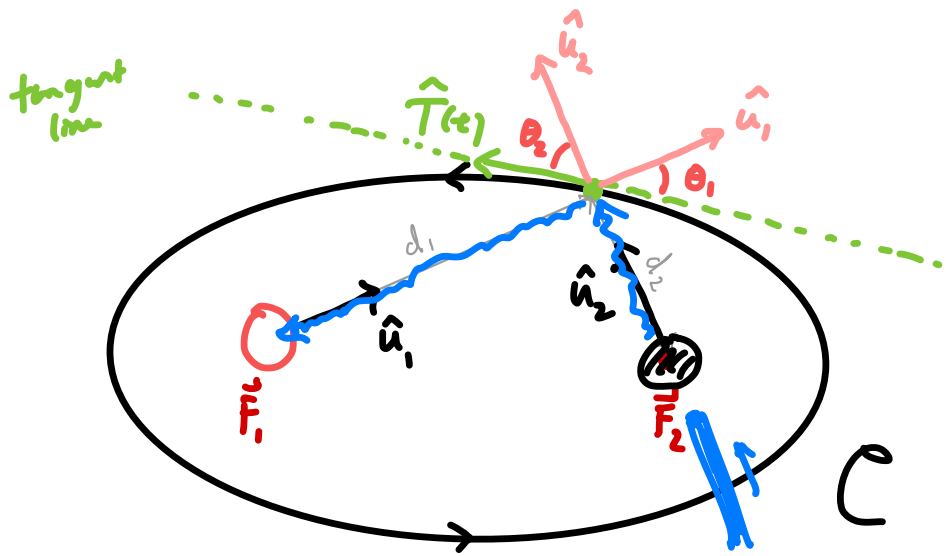
$$\hat{T}\cdot\hat{u}_1 = -\hat{T}\cdot\hat{u}_2 \implies \cos\theta_1 = \cos\theta_2 .$$

So if you ever need to sink a ball into a focal pocket on an elliptic pool table, place the ball at the other focus — it won't matter which direction you hit it!

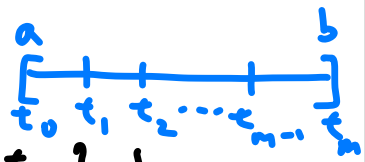
You will link this description of an ellipse to the standard one in the HW, i.e.

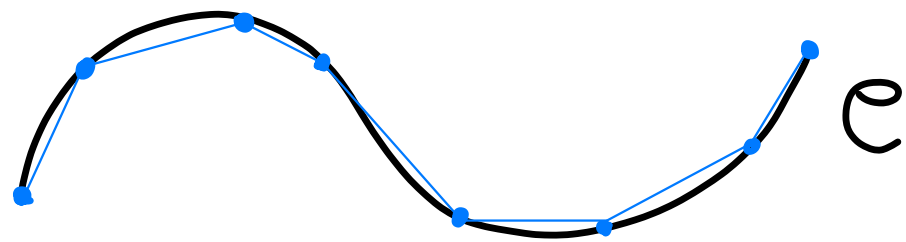
$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\ t \mapsto (a\cos t,\ b\sin t) . \end{cases}$$



§2. Arclength

Given a parametric curve $\vec{g} : \overset{[a,b]}{\overset{\shortparallel}{I}} \to \mathbb{R}^n$, we would like to measure the "length" of its image $\mathcal{C} = \vec{g}(I)$.





<u>Idea</u>: let $\mathcal{P} = \{t_0, t_1, \dots, t_m\}$ be a partition of I, and $\pi(\mathcal{P})$ the polygon with vertices $\{\vec{g}(t_0), \vec{g}(t_1), \dots, \vec{g}(t_m)\}$, with total length

$$\ell(\vec{g}, \mathcal{P}) := \sum_{k=1}^{m} \|\vec{g}(t_k) - \vec{g}(t_{k-1})\|$$

<u>Definition</u>: $\mathcal{C}$ is rectifiable if

$$\mathcal{S} := \{\ell(\vec{g}, \mathcal{P}) \mid \mathcal{P} \text{ partition of } I\}$$

is bounded above. In that event, the length of $\mathcal{C}$ is

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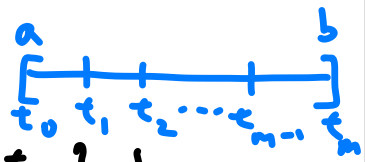

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Theorem: If $\vec{g}$ is piecewise C^1, then C is rectifiable and

$$\ell(\vec{g}) = \int_a^b \|\vec{g}'(t)\|\, dt$$

Example: $\vec{g}(t) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}$, $t \in [0, \theta] \Rightarrow$

(arc of a circle) $\vec{g}'(t) = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \end{pmatrix} \Rightarrow \|\vec{g}'\| = R$

$\Rightarrow \ell(\vec{g}) = R\theta$.

If we let b move (take $b = t$), the arclength from $\vec{g}(a)$ to $\vec{g}(t)$ is

$$s(t) := \int_a^t \|\vec{g}'(u)\|\, du,$$

and $s'(t) = \|\vec{g}'(t)\|$ (= "speed"). C is said to be parametrized by arclength if $s'(t) \equiv 1$.

(OR unit speed parametrization)

Example: (helix) $\vec{g}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 4t \end{pmatrix} \rightsquigarrow \vec{g}' = \begin{pmatrix} -3\sin t \\ 3\cos t \\ 4 \end{pmatrix}$

$$\rightsquigarrow \|\vec{g}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5$$

If $I = [0, \pi/2]$ then $\ell(\vec{g}) = \int_0^{\pi/2} 5\,dt = \frac{5\pi}{2}$.

A parametrization by arclength is given by

$$\begin{pmatrix} s(t) = 5t \\ t(s) = s/5 \end{pmatrix} \Rightarrow \vec{h}(s) = \begin{pmatrix} 3\cos(s/5) \\ 3\sin(s/5) \\ 4s/5 \end{pmatrix}.$$

Problem: Find the arclength of the curve traced out by

$$\vec{g}(t) = \begin{pmatrix} t \\ 3t^2 \\ 6t^3 \end{pmatrix} \text{ on } I = [0,2].$$

Then reparametrize it by arclength.

(just get an approximation to this part – won't get a closed formula)

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We can obtain this by writing $t(s)$ for the inverse function and replacing $\vec{g}(t)$ by $\vec{h}(s) := \vec{g}(t(s))$.

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Proof:

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$$\ell(\vec{g}, P) = \sum_{k=1}^{m} \|\vec{g}(t_k) - \vec{g}(t_{k-1})\| \underset{FTC}{=} \sum_{k=1}^{m} \left\| \int_{t_{k-1}}^{t_k} \vec{g}'(t)\,dt \right\|$$

$$\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|\vec{g}'(t)\|\,dt = \int_a^b \|\vec{g}'(t)\|\,dt$$

gives an upper bound on $\mathcal{S}$,

if $\vec{X} = \int_A^B \vec{F}(t)\,dt$

then $\|\vec{X}\|^2 = \vec{X}\cdot\vec{X} = \vec{X}\cdot\int_A^B \vec{F}(t)\,dt = \int_A^B \vec{X}\cdot\vec{F}(t)\,dt$

$$\leq \int_A^B |\vec{X}\cdot\vec{F}(t)|\,dt \underset{\text{Cauchy-Schwarz}}{\leq} \int_A^B \|\vec{X}\|\|\vec{F}(t)\|\,dt$$

$$= \|\vec{X}\| \int_A^B \|\vec{F}(t)\|\,dt$$

Now canceling $\|\vec{X}\|$ gives

$$\left\| \int_A^B \vec{F}(t)\,dt \right\| \leq \int_A^B \|\vec{F}(t)\|\,dt.$$

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$$\vec{g}(t) = \begin{pmatrix} t \\ 3t^2 \\ 6t^3 \end{pmatrix} \quad \text{on } I = [0,2].$$

Then reparametrize it by arclength.

Definition: C is rectifiable if

$$\mathcal{S} := \{\ell(\vec{g}, P) \mid P \text{ partition of } I\}$$

is bounded above. In that event, the length of C is

$$\ell(\vec{g}) := \sup\{\mathcal{S}\}$$

Theorem: If $\vec{g}$ is piecewise C^1, then C is rectifiable and

$$\ell(\vec{g}) = \int_a^b \|\vec{g}'(t)\|\,dt$$

Proof: First,

$$\ell(\vec{g}, P) = \sum_{k=1}^{m} \|\vec{g}(t_k) - \vec{g}(t_{k-1})\| \underset{FTC}{=} \sum_{k=1}^{m} \left\| \int_{t_{k-1}}^{t_k} \vec{g}'(t)\,dt \right\|$$

$$\leq \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|\vec{g}'(t)\|\,dt = \int_a^b \|\vec{g}'(t)\|\,dt$$

gives an upper bound on $\mathcal{S}$, and $\sup(\mathcal{S}) \leq$ any upper bound. So C is rectifiable, and $\ell(\vec{g}) \leq \int_a^b \|\vec{g}'(t)\|\,dt$.

Applying this to the interval $[t, t+h]$,

$$\underbrace{\|\vec{g}(t+h) - \vec{g}(t)\|}_{\ell(\vec{g},P) \text{ for trivial partition}} \leq \underbrace{\underbrace{s(t+h)}_{\text{arclength on } [a,t+h]} - \underbrace{s(t)}_{\text{arclength on } [a,t]}}_{\text{difference = arclength on } [t,t+h]} \leq \int_t^{t+h} \|\vec{g}'(t)\|\,dt$$

(assume arclength additive)

Example: (helix) $\vec{g}(t) = \begin{pmatrix} 3\cos t \\ 3\sin t \\ 4t \end{pmatrix} \rightsquigarrow \vec{g}' = \begin{pmatrix} -3\sin t \\ 3\cos t \\ 4 \end{pmatrix}$

$$\rightsquigarrow \|\vec{g}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5$$

If $I = [0, \pi/2]$ then $\ell(\vec{g}) = \int_0^{\pi/2} 5\,dt = \frac{5\pi}{2}$.

A parametrization by arclength is given by

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Applying this to the interval $[t, t+h]$,

$$\frac{\|\vec{g}(t+h) - \vec{g}(t)\|}{h} \leq \frac{s(t+h) - s(t)}{h} \leq \frac{\int_t^{t+h} \|\vec{g}'(t)\|\,dt}{h}$$

$h \to 0$

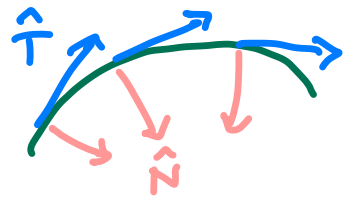
Squeeze thm.

$$\|\vec{g}'(t)\|$$

$\Rightarrow s'(t) = \|\vec{g}'(t)\| \Rightarrow s(t) = \int_a^t \|\vec{g}'(u)\|\,du.$ $\square$

§3. Curvature

Given $\vec{g}: I \to \mathbb{R}^n$, recall that $v := \|g'\|$, $\hat{T} := \frac{\vec{g}'}{\|\vec{g}'\|}$, and define the unit normal vector by $\hat{N} := \frac{\hat{T}'}{\|\hat{T}'\|}$.



Since $\vec{g}' = \|g'\| \hat{T} = v\hat{T}$, differentiating gives

$$\vec{g}'' = v'\hat{T} + v\hat{T}' = \underbrace{v'\hat{T}}_{\text{tangential}} + \underbrace{v\|\hat{T}'\|\hat{N}}_{\text{normal}}$$

tangential & normal components of acceleration.

The normal component measures how tightly we are turning. To make this measure independent of how C is parametrized, set

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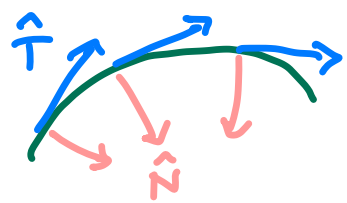

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$$\kappa(t) := \|\vec{K}(t)\| = \frac{\|\hat{T}'(t)\|}{v(t)}.$$

Example: $\vec{g}(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix} \rightsquigarrow \vec{g}' = \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix}$

$\rightsquigarrow v = \|\vec{g}'\| = R$ and $\hat{T} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

$\rightsquigarrow \hat{T}' = \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}$. So $\kappa = \frac{\|\hat{T}'\|}{v} = \frac{1}{R}$. //

Motivated by this example, we define the radius of curvature of C at $\vec{g}(t)$ by

$$R(t) := \frac{1}{\kappa(t)}.$$

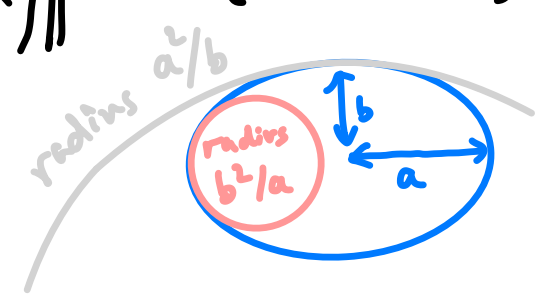
To actually compute κ for other examples, we need a better formula (proved in your HW):

$$\kappa(t) = \frac{\|\vec{g}'(t) \times \vec{g}''(t)\|}{\|\vec{g}'(t)\|^3}.$$

Example: $\vec{g}(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} \rightsquigarrow \vec{g}(t) = \begin{pmatrix} a\cos t \\ b\sin t \\ 0 \end{pmatrix}$

$$\kappa(t) = \frac{\left\| \begin{pmatrix} -a\sin t \\ b\cos t \\ 0 \end{pmatrix} \times \begin{pmatrix} -a\cos t \\ -b\sin t \\ 0 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} -a\sin t \\ b\cos t \\ 0 \end{pmatrix} \right\|^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

If $a > b$ this leads to the picture



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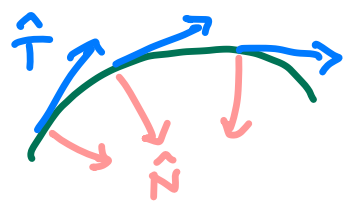

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Going back to the starred formulas, combining them gives

$$\vec{g}'' = v'\hat{T} + v^2\kappa\hat{N}.$$

Problem: Using this formula, explain how engineers might decide at what angle to bank a road that is a circle of radius $\frac{1}{4}$ mile ($=1320$ ft.) around which cars wish to drive safely at 40 mph ($= 58\frac{2}{3}$ ft./sec.). (The acceleration due to gravity is 32 ft./sec^2.)

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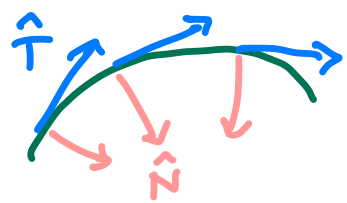

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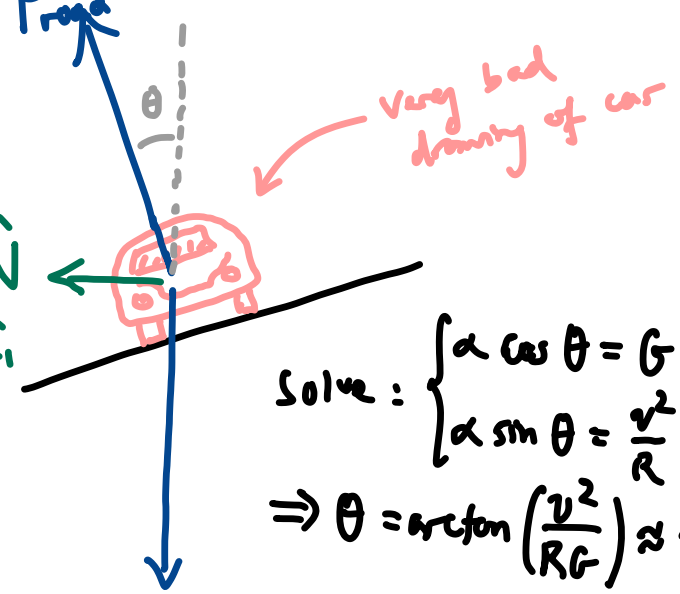
$$m\begin{bmatrix}\alpha\sin\theta\\ \alpha\cos\theta\end{bmatrix} = \vec{F}_{road}$$



$$m\vec{g}'' = mv^2\kappa\hat{N} = -m\frac{v^2}{R}\hat{e}_1$$

$\uparrow$ mass

Solve: $\begin{cases}\alpha\cos\theta = G\\ \alpha\sin\theta = \frac{v^2}{R}\end{cases}$

$$\Rightarrow \theta = \arctan\left(\frac{v^2}{RG}\right) \approx 4.66^\circ.$$

$$\vec{F}_{gravity} = -mG\hat{e}_2$$

§4. Gradients

Let $f : \mathcal{S} \longrightarrow \mathbb{R}$ be a scalar function of n variables,
$\mathcal{S} \subset \mathbb{R}^n$

$$\mathcal{L}(k) := \{\vec{x} \in \mathcal{S} \mid f(\vec{x}) = k\}$$

a level set of f,

and $\vec{a} \in \mathcal{L}(k)$ a point at which f is differentiable with <u>nonzero</u> $\vec{\nabla} f(\vec{a})$.

Define the <u>tangent plane</u> to $\mathcal{L}(k)$ at $\vec{a}$ by:

$$T_{\vec{a}}\mathcal{L}(k) := \left\{ \vec{a} + \vec{r}\,'(0) \;\middle|\; \begin{array}{l} \vec{r} : (-\epsilon, \epsilon) \to \mathcal{L}(k) \text{ diff.} \\ \text{curve with } \vec{r}(0) = \vec{a} \end{array} \right\}$$

(Here "plane" is a catch-all term — this has dimension $n-1$.) A vector is <u>perpendicular</u> to $\mathcal{L}(k)$ at $\vec{a}$ if it is perpendicular to $T_{\vec{a}}\mathcal{L}(k)$.

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Theorem: $\vec{\nabla} f(\vec{a})$ is perpendicular to $\mathcal{L}(k)$ at $\vec{a}$. Hence the equation of the tangent plane is $\vec{\nabla} f(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$, and the directional derivative $D_{\hat{u}} f(\vec{a})$ is maximized by taking $\hat{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$.

Proof: We have (for any curve as above)

$$F(t) := f(\vec{r}(t)) = k \implies 0 = F'(t)$$

$$\implies 0 = F'(0) = \vec{\nabla} f(\vec{r}(0)) \cdot \vec{r}\,'(0) = \vec{\nabla} f(\vec{a}) \cdot \vec{r}\,'(0).$$

The difference of any 2 points in $T_{\vec{a}}\mathcal{L}(k)$ is of the form $(\vec{a} + \vec{r}_1\,'(0)) - (\vec{a} + \vec{r}_2\,'(0)) = \vec{r}_1\,'(0) - \vec{r}_2\,'(0)$, which is $\perp$ to $\vec{\nabla} f(\vec{a})$.

The directional derivative is

$$(D_{\hat{u}} f)(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \hat{u} = \|\vec{\nabla} f(\vec{a})\| \cos\theta,$$

where θ is the angle between $\vec{\nabla} f(\vec{a})$ and $\hat{u}$. To maximize it, take $\theta = 0$. $\square$

§5. Kepler's 2nd Law

Let $\vec{F}(\vec{x})$ be a (vector) force field of the form $f(\vec{x})\,\vec{x}$, so that it everywhere points toward (or away from) the origin $\vec{0}$. Let $\vec{g}(t)$ be the curvilinear motion of a mass in this field, so that

$$\boxed{m\,\vec{g}''(t) = \vec{F}(\vec{g}(t)) = f(\vec{g}(t))\,\vec{g}(t)}$$

by Newton's 2nd Law. Consider the function $\vec{A}(t) := \vec{g}(t) \times \vec{g}'(t)$, with

$$\vec{A}'(t) = \underbrace{\vec{g}'(t) \times \vec{g}'(t)}_{0} + \vec{g}(t) \times \underbrace{\vec{g}''(t)}_{\frac{1}{m} f(\vec{g}(t))\,\vec{g}(t)}$$

$$= \frac{1}{m} f(\vec{g}(t))\,\underbrace{\vec{g}(t) \times \vec{g}(t)}_{0} = 0$$

$$\Rightarrow \vec{A}(t) = \vec{A}_0 \quad (\text{constant}).$$

you'll deal with $\vec{A}_0 = \vec{0}$ in HW. Assume $\vec{A}_0 \neq \vec{0}$

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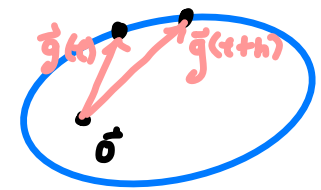
Then $\vec{A}_0 = \vec{g}(t) \times \vec{g}'(t)$ is $\perp$ to $\vec{g}(t)$

$\Rightarrow$ $\vec{g}(t)$ lies in the plane $\vec{A}_0 \cdot \vec{x} = 0$.

The area swept out from time t to $t+h$ is $\approx$ the triangle area:



$$a(t+h) - a(t) \approx \frac{1}{2}\|\vec{g}(t) \times \vec{g}(t+h)\| = \frac{1}{2}\|\vec{g}(t) \times (\vec{g}(t+h) - \vec{g}(t))\|$$

$$\Rightarrow a'(t) = \lim_{h \to 0} \frac{\frac{1}{2}\|\vec{g}(t) \times (\vec{g}(t+h) - \vec{g}(t))\|}{h}$$

rate at which area is swept out

$$= \frac{1}{2}\left\|\vec{g}(t) \times \lim_{h \to 0} \frac{\vec{g}(t+h) - \vec{g}(t)}{h}\right\|$$

$$= \frac{1}{2}\|\vec{g}(t) \times \vec{g}'(t)\|$$

$$= \frac{1}{2}\|\vec{A}_0\|$$

is constant

and we see that Kepler's 2nd Law is a special case of a very general phenomenon!