PDEs & RREFs

7

§1. Clairaut's Theorem

Let $f\begin{pmatrix}x\\y\end{pmatrix}$ be a function whose second partials $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ exist in an open set containing $\vec{0}$.

Is there a difference between

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \quad \& \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \ ?$$

By definition $f_x\begin{pmatrix}0\\y\end{pmatrix} = \lim\limits_{\Delta x \to 0} \frac{f\begin{pmatrix}\Delta x\\y\end{pmatrix} - f\begin{pmatrix}0\\y\end{pmatrix}}{\Delta x}$, so

$$f_{xy}(\vec{0}) = \lim_{\Delta y \to 0} \frac{\left(\lim\limits_{\Delta x \to 0} \frac{f\begin{pmatrix}\Delta x\\\Delta y\end{pmatrix} - f\begin{pmatrix}0\\\Delta y\end{pmatrix}}{\Delta x}\right) - \left(\lim\limits_{\Delta x \to 0} \frac{f\begin{pmatrix}\Delta x\\0\end{pmatrix} - f\begin{pmatrix}0\\0\end{pmatrix}}{\Delta x}\right)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f\begin{pmatrix}\Delta x\\\Delta y\end{pmatrix} - f\begin{pmatrix}0\\\Delta y\end{pmatrix} - f\begin{pmatrix}\Delta x\\0\end{pmatrix} + f\begin{pmatrix}0\\0\end{pmatrix}}{(\Delta x)(\Delta y)}$$

which is symmetric in x & y EXCEPT for the order of the limits.

Can we switch the order?

$\boxed{\text{PROBLEM}}$ Compute $f_x\begin{pmatrix}0\\y\end{pmatrix}$ & $f_y\begin{pmatrix}x\\0\end{pmatrix}$, then $f_{xy}\begin{pmatrix}0\\0\end{pmatrix}$ & $f_{yx}\begin{pmatrix}0\\0\end{pmatrix}$, for the function

$$f\begin{pmatrix}x\\y\end{pmatrix} := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \begin{pmatrix}x\\y\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} \\ 0, & \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \end{cases}$$

As you have just seen, we cannot necessarily swap the limits: $f_{yx}(\vec{0}) = 1$ but $f_{xy}(\vec{0}) = -1$. And the function is continuous, with continuous f_x & f_y, hence differentiable, so seems "nice".

What is going on?!

Theorem: Assume f_{xy} & f_{yx} are continuous in a neighborhood of $(0,0)$. Then $f_{xy}(\vec{0}) = f_{yx}(\vec{0})$.

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What is going on?!

Well, the full

$$f_{xy}\begin{pmatrix}x\\y\end{pmatrix} = \frac{x^6+9x^4y^2-9x^2y^4+15y^6}{(x^2+y^2)^3}$$

is NOT continuous at $\vec{0}$ (consider horizontal + vertical limits), so maybe f isn't "nice" enough.

Theorem: Assume f_{xy} & f_{yx} are continuous in a neighborhood of $(0,0)$. Then $f_{xy}(\vec{0}) = f_{yx}(\vec{0})$.

Proof: Write $\Delta(h) := f\binom{h}{h} - f\binom{h}{0} - f\binom{0}{h} + f\binom{0}{0}$, and $g(x) := f\binom{x}{h} - f\binom{x}{0}$. By the MVT,

$$(*) \quad \boxed{\frac{\Delta(h)}{h} = \frac{g(h)-g(0)}{h} = g'(a) = f_x\binom{a}{h} - f_x\binom{a}{0}}$$

for some $a \in [0,h]$. Setting $G(y) := f_x\binom{a}{y}$,

$$\frac{\Delta(h)}{h^2} \underset{(*)}{=} \frac{G(h)-G(0)}{h} \underset{\text{MVT}}{=} G'(b) = f_{xy}\binom{a}{b}$$

for some $b \in [0,h]$, and

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{\binom{a}{b}\to\binom{0}{0}} f_{xy}\binom{a}{b} = f_{xy}\binom{0}{0}$$

by continuity of f_{xy}. An exactly symmetric argument (in x & y) gives

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}\binom{0}{0}.$$ So $f_{xy}\binom{0}{0} = f_{yx}\binom{0}{0}$.

$\square$

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So $f_{xy}\binom{0}{0} = f_{yx}\binom{0}{0}$. □

§2. PDEs (Partial Differential Equations)

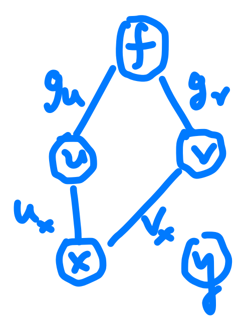
These are used to model physical situations a lot. I'll only give a brief taste here. All functions are assumed C^2 so that $f_{xy} = f_{yx}$.

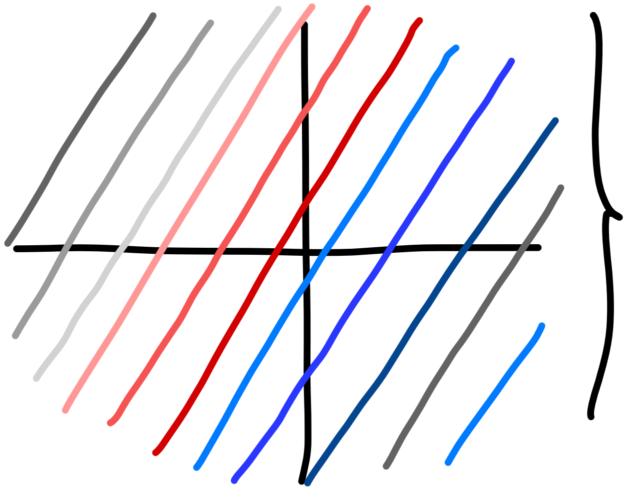
① $af_x + bf_y = 0$ (a, b real #'s)

Change coords. to $\begin{cases} u = bx - ay \\ v = ax + by \end{cases}$

Writing $f\binom{x}{y} = g\binom{u(x,y)}{v(x,y)}$ and applying the chain rule, the equation becomes

$$0 = a(\underbrace{g_u u_x + g_v v_x}_{f_x}) + b(g_u u_y + g_v v_y)$$





Solution constant on lines of slope b/a

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$$0 = a(g_u \underset{b}{\underset{\parallel}{u_x}} + g_v \underset{a}{\underset{\parallel}{v_x}}) + b(g_u \underset{-a}{\underset{\parallel}{u_y}} + g_v \underset{b}{\underset{\parallel}{v_y}})$$

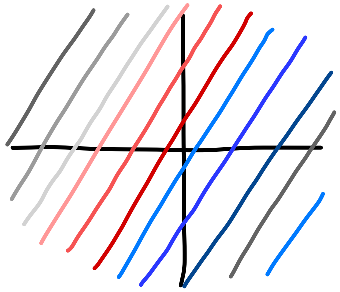
$$= \cancel{(ab - ba)} g_u + (a^2 + b^2) g_v$$

$\Longrightarrow 0 = g_v \Longrightarrow g$ constant in v

$\Longrightarrow g\begin{pmatrix} u \\ v \end{pmatrix} = G(u)$.

$\Longrightarrow f\begin{pmatrix} x \\ y \end{pmatrix} = G(u(x,y)) = G(bx - ay)$,

for some function G.



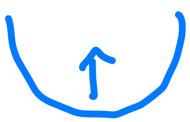
Solution constant on lines of slope b/a

① $a u_x + b u_t = 0 \Rightarrow u\binom{x}{t} = \varphi(bx - at)$

In the next example, f is a function of x (position) and t (time).

② $f_{tt} = c^2 f_{xx}$ (WAVE EQUATION)

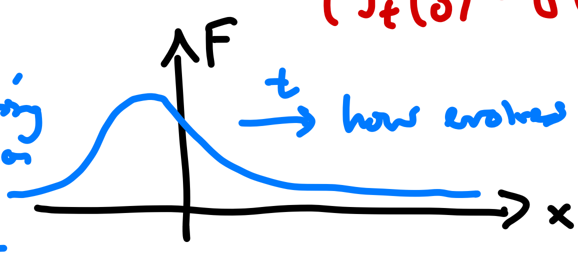
Intuition:



concavity (f_{xx}) of a string induces upward force hence acceleration (f_{tt})

W/ initial conditions
$$\begin{cases} f\binom{x}{0} := F(x) & \text{starting position} \\ f_t\binom{x}{0} := G(x) & \text{starting velocity} \end{cases}$$

starting position of wave



$\xrightarrow{t}$ how evolves?

Rewrite as

$$0 = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) f = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\overbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f}^{=:\, u\binom{x}{t}}$$

$\overset{①}{\Longrightarrow}$ $u\binom{x}{t} = \varphi(x+ct)$ for some φ.

Let $v\binom{x}{t} := \frac{1}{2c}\Phi(x+ct)$, where Φ is any antiderivative of φ. Then

$$\begin{cases} v_x = \frac{1}{2c}\Phi'(x+ct) = \frac{1}{2c}\varphi(x+ct) = \frac{u}{2c} \\ v_t = \frac{1}{2c}c\,\Phi'(x+ct) = \frac{1}{2}\varphi(x+ct) = \frac{u}{2} \end{cases}$$

$\Rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = u \Rightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(f - v) = 0$

$\overset{①}{\Longrightarrow}$ $f - v = \Psi(x - ct)$ for some Ψ

$\Longrightarrow \boxed{f\binom{x}{t} = \frac{1}{2c}\Phi(x+ct) + \Psi(x-ct)}$ (*)

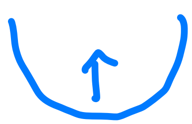
For instance, if $F(x) = \cos(kx)$ (and $G=0$),

then $f\binom{x}{t} = \frac{1}{2}\{\cos(k(x+ct)) + \cos(k(x-ct))\}$

$$= \underbrace{\cos(kx)}_{\substack{\text{original}\\ F(x)}} \underbrace{\cos(kct)}_{\substack{\text{oscillates}\\ \text{(faster for shorter waves)}}}$$

In the next example, f is a function of x (position) and t (time).

(2) $f_{tt} = c^2 f_{xx}$ (WAVE EQUATION)

Intuition:  concavity (f_{xx}) of a string induces upward force hence acceleration (f_{tt})

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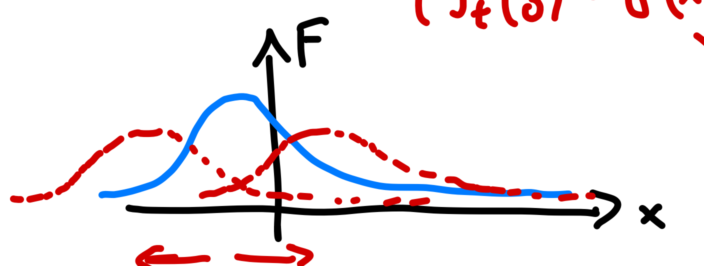
assume 0



initial wave "spreads out" to left+right at speed c

Rewrite as

$$0 = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) f = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\overbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f}^{=: u\binom{x}{t}}$$

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$\Longrightarrow$ $\boxed{f\binom{x}{t} = \frac{1}{2c}\Phi(x+ct) + \Psi(x-ct)}$ (*)

Initial conditions give

$$\begin{cases} F(x) = f\binom{x}{0} = \frac{1}{2c}\Phi(x) + \Psi(x) \\ 0 = f_t\binom{x}{0} = \frac{1}{2}\Phi'(x) - c\Psi'(x) \end{cases} \xRightarrow{\text{solve}}$$

$\Phi = cF$ and $\Psi = \frac{1}{2}F$ $\Longrightarrow$

$$\boxed{f\binom{x}{t} = \frac{1}{2}\{F(x+ct) + F(x-ct)\}}$$

For instance, if $F(x) = \cos(kx)$ (and $G=0$),

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We can generalize the wave equation to 2 space dimensions $\left(f = f\begin{pmatrix}x\\y\\t\end{pmatrix}\right)$:

$$\text{(2')} \quad f_{tt} = c^2(f_{xx} + f_{yy}) \quad (*)$$

You will prove in the HW that in polar coordinates,

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ becomes $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

So if f depends only on $r = \sqrt{x^2+y^2}$ & t,

$f_{xx} + f_{yy} = f_{rr} + \frac{1}{r} f_r$.

If f factors as a product $F(r)\,G(t)$,

$(*)$ becomes

$$FG'' = c^2\left(F'' + \tfrac{1}{r}F'\right)G$$

$$\Rightarrow \underbrace{\frac{G''}{G}}_{\text{constant in } r} = c^2 \underbrace{\frac{F'' + \frac{1}{r}F'}{F}}_{\text{constant in } t}$$

& $\Rightarrow$ constant $=: \alpha$

$$\Rightarrow G'' = \alpha G, \qquad F'' + \tfrac{1}{r}F' + F = 0$$

say $\alpha = -c^2$

$G =$ sin + cos type function $\qquad F =$ Bessel function

what happens when you throw a stone in a pond

On the other hand, what if I want no oscillation — a solution constant in time? You need to solve

$$\text{(3)} \quad 0 = F_{xx} + F_{yy} \quad \text{(Laplace's equation)}$$

and take $f\begin{pmatrix}x\\y\\t\end{pmatrix} = F\binom{x}{y}$. Solutions to (3) are called harmonic functions.

§3. Linear systems

Let A be an $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$ a vector. How to solve

$$\boxed{A\vec{x} = \vec{b}}$$

for $\vec{x} \in \mathbb{R}^n$? Write it out

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

as a system of m linear equations in n variables, and perform "Gaussian elimination", otherwise known as row-reduction.

Elementary Row Operations (EROs)

ⓐ REPLACE an equation/row by {itself + multiples of other equations/rows}

ⓑ SWAP two equations/rows

ⓒ SCALE an equation/row (i.e., multiply it by a nonzero constant)

produces a new "row-equivalent" system of equations whose solution set certainly includes all the old solutions. In fact, since ⓐ–ⓒ are reversible, the new solution set is the same:

Row-equivalent systems are equivalent.

Example: find the solution set in $\mathbb{R}^3$ of

$$\left.\begin{array}{ll} (R_1) & x_1 + x_2 + x_3 = 9 \\ (R_2) & 2x_1 + 4x_2 - 3x_3 = 1 \\ (R_3) & 3x_1 + 6x_2 - 5x_3 = 0 \end{array}\right\} \xrightarrow{\text{shorthand}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

eliminate x_1 in R_2, R_3: $R_2 \mapsto R_2 - 2R_1$, $R_3 \mapsto R_3 - 3R_1$

$$\left.\begin{array}{ll} (R_1) & x_1 + x_2 + x_3 = 9 \\ (R_2) & 2x_2 - 5x_3 = -17 \\ (R_3) & 3x_2 - 8x_3 = -27 \end{array}\right\} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array}\right]$$

eliminate x_2 in R_3: $R_3 \mapsto R_3 - \frac{3}{2}R_2$

$$\left.\begin{array}{ll} (R_1) & x_1 + x_2 + x_3 = 9 \\ (R_2) & 2x_2 - 5x_3 = -17 \\ (R_3) & -\frac{1}{2}x_3 = -\frac{3}{2} \end{array}\right\} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 0 & -\frac{1}{2} & -27 \end{array}\right]$$

So $x_3 = 3$, and back-substituting in (R_2) gives $2x_2 - 15 = -17 \Rightarrow x_2 = -1$, whereupon substituting in (R_1) gives $x_1 - 1 + 3 = 9 \Rightarrow x_1 = 7$.

You can check that $(7, -1, 3)$ solves the original system.

Why does this work? Applying

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Row-equivalent systems are equivalent.

Here's an example that uses all three operations.

Example: find the solution set in $\mathbb{R}^4$ of

$$\left.\begin{aligned} x_3 - x_4 &= -1 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 &= 2 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 &= 3 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 &= 6 \end{aligned}\right\} \to \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 2 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{\text{(b)}: R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 2 & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{2}R_1]{\text{(c)}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \mapsto R_3 - 2R_1 \\ R_4 \mapsto R_4 - 3R_1}]{\text{(a)}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{array}\right] \xrightarrow[\substack{R_3 \mapsto R_3 - R_2 \\ R_4 \mapsto R_4 - 3R_2}]{\text{(a)}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

STOP. The 3rd line corresponds to the equation $0 = 2$, and so the system is inconsistent (no solutions).

§4. The reduction algorithm

Using the EROs, I now explain how to put *any* matrix in a particularly nice form:

Reduced Row Echelon Form (RREF)

A matrix A is in RREF if all of the following hold:

(i) the 1st nonzero entry of each row is 1, called a 'leading 1';

(ii) when a column contains a leading 1, all other entries in that column are 0 (this is called a "pivot column"); and

(iii) when a row contains a leading 1, each row above it contains a leading 1 further to the left.

The weaker notion of "row-echelon form" is obtained by dropping (i) and weakening (ii) (only the entries below a leading nonzero entry must be 0).

Elementary Row Operations (EROs)

ⓐ REPLACE an equation/row by {itself + multiples of other equations/rows}

ⓑ SWAP two equations/rows

ⓒ SCALE an equation/row (i.e., multiply it by a nonzero constant)

Example: If "$*$" stands for "arbitrary numbers", then
$$\begin{pmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
are (in) RREF, while (if "$\bullet$" is "arbitrary NONZERO number"),
$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \end{pmatrix}$$
are (in) REF.

Problem: Which are in RREF (or REF)? Use EROs to put ones which aren't in RREF.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$

Row-reduction means "some procedure for associating an RREF matrix to a given matrix A via EROs".

$$A \longmapsto \mathrm{rref}(A).$$

Since the procedure simply applies EROs to A, the new matrix is row-equivalent to A.

FACT: There is exactly one RREF matrix row-equivalent to a given matrix A.

(I'll explain why next week.)

CONSEQUENCE: $\mathrm{rref}(A)$ is independent of the procedure used!

Here is one such procedure: I'd suggest using it when you don't immediately see a shortcut.

Example: $A = \begin{pmatrix} \boxed{0} & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$

$$\xrightarrow{(c)} \begin{pmatrix} \boxed{2} & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{(e)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

(b)

$$\xrightarrow{(d)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix} = \operatorname{rref}(A)$$

(f)

Row-reduction algorithm

(a) "cursor" starts at upper left-hand entry of A

(b) move cursor to the right if the cursor entry & all entries below it are 0; repeat until this is no longer the case

(c) **SWAP** if cursor entry = 0, swap cursor row with the first row below it having nonzero entry in the cursor column

(d) **SCALE** divide cursor row by cursor entry

(e) **REPLACE** eliminate all other entries in the cursor column by adding suitable multiples of the cursor row to other rows

(f) if cursor is at bottom right, STOP. Otherwise, move down and to the right, & go back to (b).

Example: $A = \begin{pmatrix} \boxed{0} & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$

$$\xrightarrow{(c)} \begin{pmatrix} \boxed{2} & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$$

$$\xrightarrow{(e)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{2} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$$\xrightarrow{(d)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix} = \text{rref}(A)$$

Now how do we use this to solve a linear system?

STEP 1 Convert the system to an augmented matrix

$$M = [A \mid \vec{b}].$$

STEP 2 Apply the row-reduction algorithm to compute

$$\text{rref}(M).$$

STEP 3 Convert back to a linear system and find the tuples $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ solving it. More precisely, use the non-pivot — or "free" — variables to parametrize the solutions.

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Example:

$$\left.\begin{aligned} 3x_1 - 6x_2 + 2x_3 - x_4 &= 1 \\ -2x_1 + 4x_2 + x_3 + 3x_4 &= 4 \\ x_3 + x_4 &= 2 \\ x_1 - 2x_2 + x_3 &= 1 \end{aligned}\right\}$$

STEP 1

$$M = \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

STEP 2

$$\mathrm{rref}(M) = \begin{array}{c} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \end{array}$$

pivot columns

free variables

STEP 3

$$\left.\begin{aligned} x_1 - 2x_2 - x_4 &= -1 \\ x_3 + x_4 &= 2 \end{aligned}\right]$$

solve for pivot variables in terms of free ones (which can be freely chosen!)

$$\begin{cases} x_1 = -1 + 2x_2 + x_4 \\ x_3 = 2 - x_4 \end{cases}$$

Finally, write out the general solution vector:

free variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 + 2x_2 + x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\text{this is called standard form.}}$$