LINEAR SYSTEMS

8

§1. A geometric viewpoint

As a warm-up, let's start with systems of n linear equations in n variables, where $n = 2$ or 3.

These will have 3 presentations:

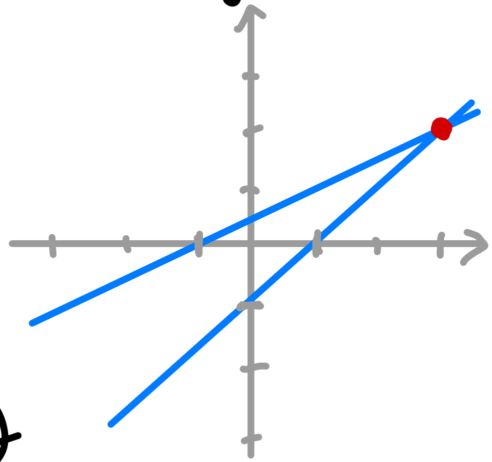
(a) by "row" equations

(b) by "column" equations

(c) by "matrix" equation.

Ex 1 (a) Consider the system

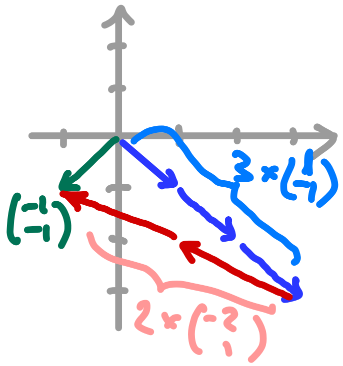
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + x_2 = -1 \end{cases}$$



with accompanying picture exhibiting $(x_1, x_2) = (3,2)$ as the unique solution.

(b) Now we can rewrite the system as

$$x_1\begin{pmatrix}1\\-1\end{pmatrix} + x_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix},$$



which asks the question:

Can we produce $\begin{pmatrix}-1\\-1\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\-1\end{pmatrix}$ & $\begin{pmatrix}-2\\1\end{pmatrix}$?

The answer, as shown, is YES.

(c) The matrix form of the equation is

$$\begin{pmatrix}1 & -2\\-1 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$

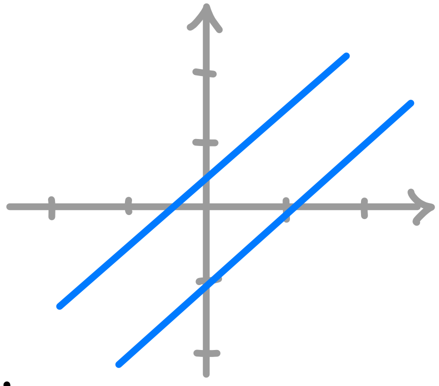
$$A \cdot \vec{x} = \vec{b}$$

//

The system in the example is called consistent because a solution exists. Here is an inconsistent one:

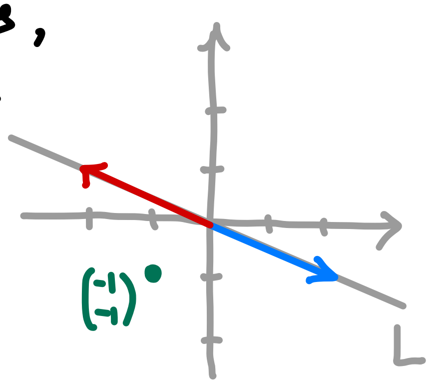
Ex 2 Tweaking the 1st eqn. in Ex. 1 we have

(a) $\begin{cases} 2x_1 - 2x_2 = -1 \\ -x_1 + x_2 = -1 \end{cases}$



which corresponds to a pair of parallel lines, while in (b) we have

$$x_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$



which is impossible (as any linear combination of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ lie on the line L, and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ does not) //

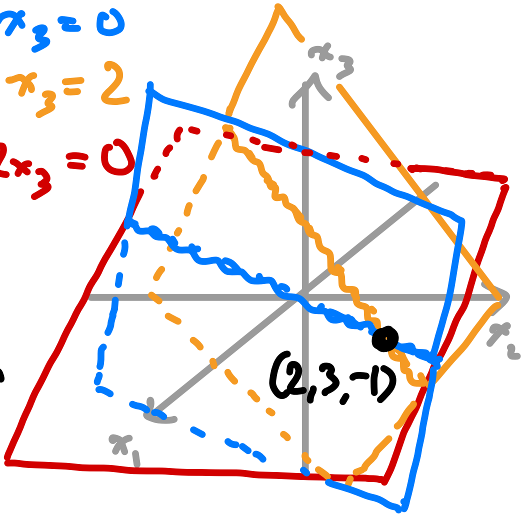
Ex 3 Finally, if we change $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ to something lying on L, say $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$, then there are <u>many</u> linear combinations of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ that will do (for (b)). Correspondingly, the two parallel lines in (a) <u>merge</u>, and we have <u>infinitely many solutions</u> (⇒ consistent). //

Turning to n=3, here is a consistent system:

Ex 4 **ROW**

(a) $\begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_3 = 2 \\ x_1 \qquad + 2x_3 = 0 \end{cases}$

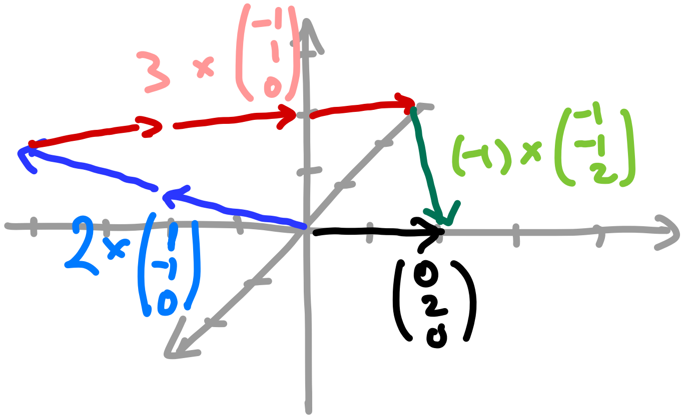
3 planes in $\mathbb{R}^3$



Note that two of the planes pass through the origin (0,0,0).

COLUMN

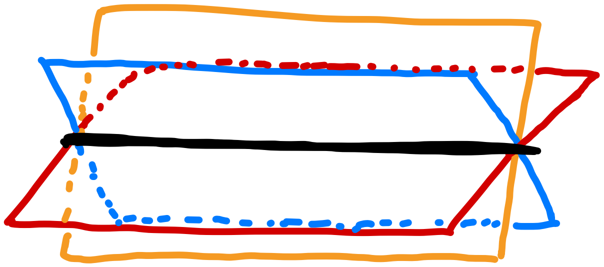
(b) $x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$



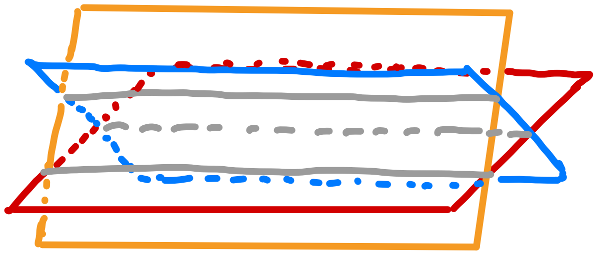
MATRIX

(c) $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$. //

Ex 5 If in Ex. 4(a) we made the last equation $x_3 = -1$, we would get a line as solution set:



On the other hand, if we move the $x_3 = -1$ plane up or down to $x_3 = a$ $(a \neq -1)$, then we get the picture



so that there are no common solutions (and the system is inconsistent).

Correspondingly, the linear combinations $x_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + x_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix}$ include $\begin{pmatrix}0\\2\\-1\end{pmatrix}$, but **NOT** $\begin{pmatrix}0\\2\\a\end{pmatrix}$ for $a \neq -1$. //

From these examples we can (informally) glean that

(i) There are 3 possibilities for linear systems: no solutions, one solution, or infinitely many

(ii) The n equations have a common solution (i.e. are consistent) $\Longleftrightarrow$ the column vector on the RHS of the vector equation can be written as a linear combination of the column vectors on the LHS.

$A\vec{x} = \vec{b}$

(iii) The equations have a common solution for every $\vec{b}$ $\Longleftrightarrow$ linear combinations of the LHS column vectors fill up all of n-space.

A little more precisely & generally:

$$A\vec{x} = \vec{b} \quad \overset{\text{rows}}{\longleftrightarrow} \quad \begin{pmatrix} \leftarrow \vec{A}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{A}_m \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow \\ \vec{x} \\ \downarrow \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$\Big\updownarrow$$

$$\begin{cases} \vec{A}_1 \cdot \vec{x} = b_1 \\ \quad\vdots \\ \vec{A}_m \cdot \vec{x} = b_m \end{cases}$$

columns

$$\Big\updownarrow$$

$$\begin{pmatrix} \uparrow & & \uparrow \\ \vec{a}_1 & \cdots & \vec{a}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \uparrow \\ \vec{b} \\ \downarrow \end{pmatrix}$$

$$\Big\updownarrow$$

$$\Big\{ x_1 \begin{pmatrix} \uparrow \\ \vec{a}_1 \\ \downarrow \end{pmatrix} + \cdots + x_n \begin{pmatrix} \uparrow \\ \vec{a}_n \\ \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow \\ \vec{b} \\ \downarrow \end{pmatrix}$$

vector equation

has a solution in $\vec{x}$ (is consistent) for every $\vec{b}$

$$\Updownarrow$$

$$\vec{b} \in \operatorname{span}\{\vec{a}_1, \dots, \vec{a}_n\}$$

i.e. you can write $\vec{b}$ as a linear combination of the columns of A.

§2. Review of row-reduction

How do we check whether a system of linear equations is consistent, or (equivalently) a vector $\vec{b}$ is in the span of some other vectors $\vec{a}_1, \dots, \vec{a}_n$?

STEP 1 Convert the system to an augmented matrix

$$M = [A \mid \vec{b}].$$

STEP 2 Apply the row-reduction algorithm to compute

$$\operatorname{rref}(M).$$

STEP 3 Convert back to a linear system and find the tuples $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ solving it. More precisely, use the non-pivot — or "free" — variables to parametrize the solutions.

For STEP 2, we use

Elementary Row Operations (EROs)

ⓐ REPLACE an equation/row by {itself + multiples of other equations/rows}

ⓑ SWAP two equations/rows

ⓒ SCALE an equation/row (i.e., multiply it by a nonzero constant)

to put $[A \mid \vec{b}]$ in

Reduced Row Echelon Form (RREF)

A matrix A is in RREF if all of the following hold:

(i) the 1st nonzero entry of each row is 1, called a 'leading 1';

(ii) when a column contains a leading 1, all other entries in that column are 0 (this is called a "pivot column"); and

(iii) when a row contains a leading 1, each row above it contains a leading 1 further to the left.

The new system has the same solution set as the old one.

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From this Example, we see that

- If the last column of the augmented matrix is a pivot column, then the system is inconsistent.

Otherwise, Steps 1-3 tell us how to solve the system, and so:

- If the last column is non-pivot, the system is consistent; it has a unique solution if there are no free variables, i.e. if all but the last column are pivots.

So far we have looked at m equations in n unknowns, with $m=n$. What about the other cases?

- If $m>n$, the system is called overdetermined. If $m<n$, it is underdetermined.

An overdetermined system can be consistent, but some equations will have to be "linear combinations" of others.

An underdetermined system cannot have a unique solution: think about intersections of two planes in $\mathbb{R}^3$ — these will never intersect in a point.

Ex 1 Is $\begin{pmatrix}1\\4\\3\\1\end{pmatrix}$ in Span$\left\{\begin{pmatrix}3\\-2\\0\\1\end{pmatrix},\begin{pmatrix}-6\\4\\0\\-2\end{pmatrix},\begin{pmatrix}2\\1\\1\\1\end{pmatrix},\begin{pmatrix}-1\\3\\1\\0\end{pmatrix}\right\}$?

ANSWER: NO! Because the last column is a pivot column.

$$\begin{array}{cccc|c} & A & & & \vec{b} \\ \hline 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 & 1 \end{array}$$

$$\Big\downarrow \text{ Scale } R_1 \mapsto \tfrac{1}{3}R_1$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{ Replace } R_2 \mapsto R_2+2R_1,\ R_4 \mapsto R_4-R_1$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 7/3 & 7/3 & 14/3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right]$$

$$\Big\downarrow \text{ Scale } R_2 \mapsto \tfrac{3}{7}R_2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right]$$

$$\Big\downarrow \text{ Replace } R_3 \mapsto R_3-R_2,\ R_4 \mapsto R_4-\tfrac{1}{3}R_2,\ R_1 \mapsto R_1-\tfrac{2}{3}R_2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow \text{ Replace } R_1 \mapsto R_1+R_3,\ R_2 \mapsto R_2-2R_3$$

$$\underbrace{\left[\begin{array}{cccc|c} \boxed{1} & -2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{rref}[A|\vec{b}]}$$

pivot columns

//

From this Example, we see that

- If the last column of the *augmented matrix* is a pivot column, then the system is inconsistent.

Otherwise, Steps 1-3 tell us how to solve the system, and so:

- If the last column is non-pivot, the system is consistent; it has a *unique* solution if there are no free variables, i.e. if *all but* the last column are pivots.

So far we have looked at m equations in n unknowns, with $m=n$. What about the other cases?

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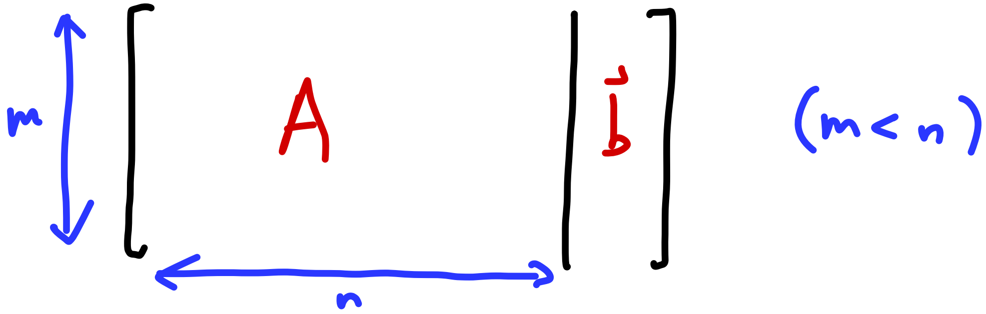
We can make this more precise: when you reduce an augmented matrix of the form



to RREF, there *must* be non-pivot columns, hence free variables.

This is because each pivot column contains a leading 1 for some row, and there are only m rows hence $\leq m$ (thus $<n$) leading 1's.

In general, *the number of pivots is at most the smaller of m & n*.

This number is called the *rank* of a matrix.

§3. Linear combinations

(A $m \times n$)

We have seen that

① $A\vec{x}=\vec{b}$ has a solution

is equivalent to

② $\vec{b} \in \text{span}\{\text{columns of } A\}$.

Here is another set of equivalent assertions:

Ⓐ span{columns of A} = (all of) $\mathbb{R}^m$ $\updownarrow$ clear

Ⓑ $A\vec{x}=\vec{b}$ is consistent for any $\vec{b}$ $\updownarrow$ why?

Ⓒ rref(A) has no rows of all zeroes

Ⓓ rank(A) = m

every row has a leading 1 ⟺ m pivots

To understand Ⓑ⟺Ⓒ, note:

- $[A\,|\,\vec{b}] \underset{\text{equiv.}}{\overset{\text{row-}}{\sim}} [\text{rref}(A)\,|\,\vec{c}]$ for some $\vec{c} \in \mathbb{R}^m$

  (apply the sequence of row operations that put A in rref, to the augmented matrix)

- We can choose $\vec{b}$ so that $\vec{c}$ is any vector (because row operations are reversible)

Ⓒ ⟹ Ⓑ: If Ⓒ holds, $[\text{rref}(A)\,|\,\vec{c}]$ has a leading '1' in every row in the "rref" part, hence is in RREF itself and equals $\text{rref}[A\,|\,\vec{b}]$. Since the leading 1's occur to the left of $\vec{c}$, $\vec{c}$ is not a pivot column and the system is consistent (regardless of $\vec{b}$).

Ⓑ ⟹ Ⓒ: If Ⓒ fails, choose $\vec{b}$ so that $\vec{c}$ has a nonzero last entry. Since the last row of rref(A) is all zeroes, $\vec{c}$ is a pivot column (for this choice of $\vec{b}$), and so Ⓑ fails.

$\boxed{\text{Ex 1}}$ For which $\vec{b}$ is $\begin{pmatrix}3 & -1\\ -9 & 3\end{pmatrix}\vec{x}=\vec{b}$ solvable?

(Equivalently: determine span$\left\{\begin{pmatrix}3\\-9\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix}\right\}$.)

$$\left[\begin{array}{cc|c}3 & -1 & b_1\\ -9 & 3 & b_2\end{array}\right] \to \left[\begin{array}{cc|c}1 & -1/3 & b_1/3\\ -9 & 3 & b_2\end{array}\right] \to \left[\begin{array}{cc|c}1 & -1/3 & b_1/3\\ 0 & 0 & b_2+3b_1\end{array}\right]$$

Ⓒ fails

So we must have $\underbrace{b_2+3b_1=0}_{\text{constraint equation.}}$

//

PROBLEMS

① Calculate the rank of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$.
Do its columns span $\mathbb{R}^3$?

- $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(A) = 2$
- NO

② Suppose A is a 4×4 matrix, and $\vec{b} \in \mathbb{R}^4$ are such that $A\vec{x} = \vec{b}$ has a unique solution. Must columns of A span $\mathbb{R}^4$?

- taking rref $[A \mid \vec{b}]$, the last column must be non-pivot (for existence of solution); and there can be no non-pivot columns ← free variables in the "$[\cdots \mid$" part (for uniqueness).
- So all columns of A are pivot & A 4×4
  $\Rightarrow$ all rows have a leading '1'
  $\Rightarrow$ no rows of 0's
  $\Rightarrow$ columns span $\mathbb{R}^4$.

By the algorithm, $A \sim \text{rref}(A) =: U$.
If $A \sim V$ (also RREF), then $U \sim V$.
So if (†) is known, we get $U = V$ as desired.

§4. Uniqueness of RREF

Last time I claimed the

Theorem: Every matrix A is row-equivalent to a unique RREF matrix.

Without this, our notions of "pivot columns", "rank", and "rref(A)" aren't well-defined.

Write "$A \sim B$" for "A is row-equivalent to B".

Notice that

(*) $A \sim B \implies$ rows of A are linear combinations of rows of B (& vice-versa).

This is b/c row-operations replace a given row by a linear combination of rows, and are reversible.

To prove the Theorem, proceed in three steps:

Step 1 It is enough to show that

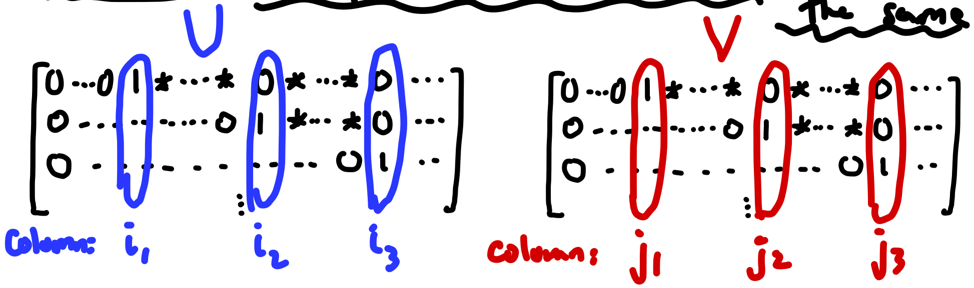
(†) 2 row-equivalent RREF matrices must be the same.

Why?

By the algorithm, $A \sim \mathrm{rref}(A) =: U$.
If $A \sim V$ (also RREF), then $U \sim V$.
So if (†) is known, we get $U = V$ as desired.

For the remaining 2 steps, let $U \sim V$ be two RREF matrices.

Step 2 The pivot columns of U & V are the same.



- Write $\vec{U}_1, \vec{U}_2, \dots$ for rows of U, similar for V.

By (*), $\vec{U}_1 \in \mathrm{span}\{\text{rows of } V\} \Rightarrow$

$$\vec{U}_1 = a_1\vec{V}_1 + a_2\vec{V}_2 + \cdots \Rightarrow i_1 \geq j_1 .$$

Reversing U & V gives $i_1 \leq j_1$. So $i_1 = j_1$ (& $a_1 = 1$).

- Next, $\vec{U}_2 = b_1\vec{V}_1 + b_2\vec{V}_2 + \cdots \Rightarrow$
$\vec{U}_2$ has b_1 in i_2^{th} entry $\Rightarrow b_1 = 0$
$\Rightarrow \vec{U}_2 = b_2\vec{V}_2 + b_3\vec{V}_3 + \cdots$ (& vice versa).

§4. Uniqueness of RREF

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Theorem: Every matrix A is row-equivalent to a unique RREF matrix.

- Striking out the first row of U & V, the remaining (still RREF!) matrices are row-equivalent. Go back to the first bullet, find $i_2 = j_2$, & repeat until nothing's left.

Step 3 The non-pivot columns of U&V are $=$.

- We have $\vec{U}_1 = \vec{V}_1 + a_2\vec{V}_2 + \cdots + a_m\vec{V}_m$
$\Rightarrow \vec{U}_1$ has a_2 in the $(j_2=)\,i_2^{\text{th}}$ coordinate
a_3 . . . $(j_3=)\,i_3^{\text{th}}$ - - - etc.
$\Rightarrow 0 = a_2 = a_3 = \cdots = a_m$. So $\vec{U}_1 = \vec{V}_1$.

- Strike out the 1st rows, apply this to the second rows to get $\vec{U}_2 = \vec{V}_2$, etc.

□

§5. Elementary matrices

Recall the 3 types of row operations on a matrix: replace, swap, scale.

I claim that these can be achieved by multiplying A by "elementary matrices".

$\boxed{\text{Ex 1}}$ Some examples with 2×2 & 3×3:

replace

$$\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-1&0\\-2&2&1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&-1&0\\0&0&1\\0&1&-1\end{pmatrix}$$

2× +

swap

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&-1&0\\0&0&1\\0&1&-1\end{pmatrix}=\begin{pmatrix}1&-1&0\\0&1&-1\\0&0&1\end{pmatrix}$$

scale

$$\begin{pmatrix}1/a&0\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}1&b/a\\c&d\end{pmatrix}$$

$\cdot\frac{1}{a}$

Main idea: row-vector times A takes a linear combination of its rows. //

More generally.

$$\begin{array}{cc}
i\text{th row}\rightarrow & \begin{pmatrix}1&&&\\&\ddots&&a\\&&\ddots&\\&&&1\end{pmatrix}\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\vdots\\\leftarrow\vec{A}_i\rightarrow\\\vdots\\\leftarrow\vec{A}_j\rightarrow\\\vdots\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}=\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\vdots\\\leftarrow\vec{A}_i+a\vec{A}_j\rightarrow\\\vdots\\\leftarrow\vec{A}_j\rightarrow\\\vdots\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}\\
 & \uparrow\ j\text{th column}
\end{array}$$

$$\begin{array}{cc}
 & \\
i & \\
j &
\end{array}\begin{pmatrix}1&&&&&&\\&\ddots&&&&&\\&&1&&&&\\&&&0&\ 1&&\\&&&1&\ 0&&\\&&&&&\ddots&\\&&&&&&1\end{pmatrix}\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\vdots\\\leftarrow\vec{A}_i\rightarrow\\\vdots\\\leftarrow\vec{A}_j\rightarrow\\\vdots\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}=\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\leftarrow\vec{A}_j\rightarrow\\\leftarrow\vec{A}_i\rightarrow\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}$$

$i \quad j$

$$\begin{pmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&&\alpha&&\\&&&&1\\&&&&&\ddots\\&&&&&&1\end{pmatrix}\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\vdots\\\leftarrow\vec{A}_i\rightarrow\\\vdots\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}=\begin{pmatrix}\leftarrow\vec{A}_1\rightarrow\\\vdots\\\leftarrow\alpha\vec{A}_i\rightarrow\\\vdots\\\leftarrow\vec{A}_m\rightarrow\end{pmatrix}$$

The elementary matrices are the 3 kinds of matrices on the left, producing Replace, Swap, & Scale operations. They are m×m.

$\boxed{\text{Ex 2}}$ Consider the row-reduction from our earlier example:

$$\begin{array}{cccc|c} & & A & & \vec{b} \\ 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 & 1 \end{array}$$

$\downarrow$ Scale $R_1 \mapsto \frac{1}{3}R_1$

$$E_1 = \begin{pmatrix} 1/3 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

$\downarrow$ Replace $R_2 \mapsto R_2 + 2R_1$, $R_4 \mapsto R_4 - R_1$

$$E_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & & & 1 \end{pmatrix} \cdot E_2 = \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 7/3 & 7/3 & 14/3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right]$$

$\downarrow$ Scale $R_2 \mapsto \frac{3}{7}R_2$

$$E_4 = \begin{pmatrix} 1 & & & \\ & 3/7 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 2/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1/3 & 1/3 & 2/3 \end{array}\right]$$

$\downarrow$ Replace $R_3 \mapsto R_3 - R_2$, $R_4 \mapsto R_4 - \frac{1}{3}R_2$, $R_1 \mapsto R_1 - \frac{2}{3}R_2$

$$E_7 = \begin{pmatrix} 1 & -\frac{2}{3} & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot E_6 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & -\frac{1}{3} & & 1 \end{pmatrix} \cdot E_5 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -1 & 1 & \\ & & & 1 \end{pmatrix} \cdot$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow$ Replace $R_1 \mapsto R_1 + R_3$, $R_2 \mapsto R_2 - 2R_3$

$$E_9 = \begin{pmatrix} 1 & & & \\ & 1 & -2 & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot E_8 = \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot columns

$\text{rref}[A|\vec{b}]$

The upshot is that

$$\text{rref}[A|\vec{b}] = E_9 \cdots E_2 E_1 [A | \vec{b}]. \quad //$$

More generally,

The result of any sequence of EROs on any matrix A can be expressed as

$$E_N \cdot \cdots \cdot E_1 \cdot A$$

where the E_i are elementary matrices.

§6. Homogeneous vs. Inhomogeneous

Homogeneous linear systems are those of the form

$$A\vec{x} = \vec{0}.$$

They are always consistent b/c $\vec{x}=\vec{0}$ is a solution. The solution set is always the span of a set of vectors:

Ex 1 Solve $A\vec{x}=\vec{0}$, where $A = \begin{pmatrix}2&2&4\\-4&-4&-8\\0&-3&-3\end{pmatrix}$.

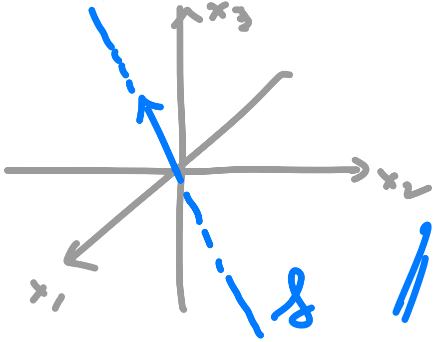
$$\left[\begin{array}{ccc|c}2&2&4&0\\-4&-4&-8&0\\0&-3&-3&0\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}\boxed{1}&0&1&0\\0&\boxed{1}&1&0\\0&0&0&0\end{array}\right]$$

x_3 free

$$\Rightarrow \begin{cases}x_1 = -x_3\\ x_2 = -x_3\end{cases} \Rightarrow \vec{x} = \begin{pmatrix}-x_3\\-x_3\\x_3\end{pmatrix} = x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix},$$

and the solution set is

$$S = \text{span}\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}.$$



Inhomogeneous linear systems are systems

$$A\vec{x} = \vec{b} \quad \text{with } \vec{b} \neq \vec{0}.$$

In this case, $\vec{x}=\vec{0}$ is NEVER a solution. So the solution set can't be a span. If nonempty, it will be a "parallel translate" of a span:

Ex 2 Solve $A\vec{x} = \begin{pmatrix}8\\-16\\12\end{pmatrix}$ (same A).

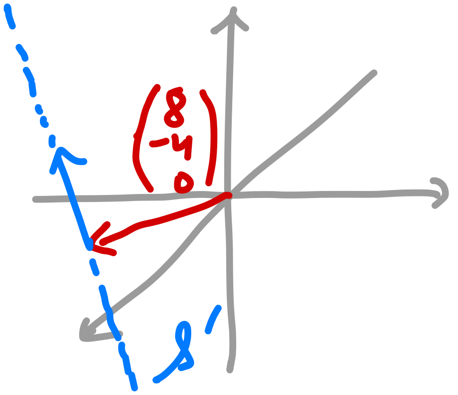
$$\left[\begin{array}{ccc|c}2&2&4&8\\-4&-4&-8&-16\\0&-3&-3&12\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c}\boxed{1}&0&1&8\\0&\boxed{1}&1&-4\\0&0&0&0\end{array}\right]$$

x_3 free

$$\Rightarrow \begin{cases}x_1 = 8 - x_3\\ x_2 = -4 - x_3\end{cases} \Rightarrow \vec{x} = \begin{pmatrix}8-x_3\\-4-x_3\\x_3\end{pmatrix} = \begin{pmatrix}8\\-4\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$\Rightarrow$ the solution set is

$$S' = \begin{pmatrix}8\\-4\\0\end{pmatrix} + \text{span}\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}.$$



The correspondence is always thus:

Suppose $A\vec{x} = \vec{0}$ has solution set $\mathcal{S}$, and $A\vec{x} = \vec{b}$ is consistent, with particular solution $\vec{x}_0$. Then its solution set is $\vec{x}_0 + \mathcal{S}$:

$$\begin{aligned} A\vec{v} = \vec{b} &\iff A(\vec{v} - \vec{x}_0) = \vec{0} \\ &\iff \vec{v} - \vec{x}_0 \in \mathcal{S} \\ &\iff \vec{v} \in \vec{x}_0 + \mathcal{S}. \end{aligned}$$

So the following are equivalent:

- the solution $\vec{x}_0$ is unique
- $\mathcal{S} = \{\vec{0}\}$
- $\overset{m \times n}{A}$ has no nonpivot (free) columns
- $\operatorname{rank}(A) = n$.

If A is square ($n \times n$) of rank n, then $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$. A is called nonsingular.