INDEPENDENCE

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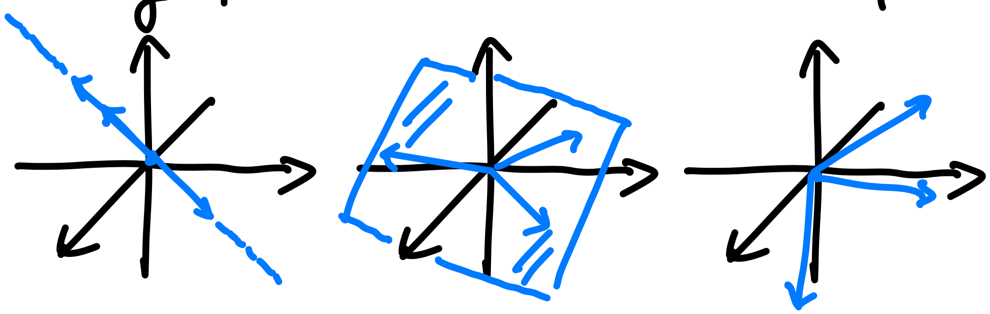
§1. Linear Independence

Ex 1 Consider the set of vectors

$$\vec{v}_1 = \begin{pmatrix}1\\4\\7\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}2\\5\\8\end{pmatrix}, \text{ and } \vec{v}_3 = \begin{pmatrix}3\\6\\9\end{pmatrix} \text{ in } \mathbb{R}^3.$$

Do they span a line, plane, or all of space?



To find out, we could ask "for what kinds of $\vec{b}$ is

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_1\begin{pmatrix}1\\4\\7\end{pmatrix} + x_2\begin{pmatrix}2\\5\\8\end{pmatrix} + x_3\begin{pmatrix}3\\6\\9\end{pmatrix} = \vec{b}$$

consistent?"

Writing this out as an augmented matrix and row-reducing yields

$$\left[\begin{array}{ccc|c}1&2&3&b_1\\4&5&6&b_2\\7&8&9&b_3\end{array}\right] \to \left[\begin{array}{ccc|c}1&2&3&b_1\\0&-3&-6&b_2-4b_1\\0&-6&-12&b_3-7b_1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&2&3&b_1\\0&1&2&-\frac{1}{3}b_2+\frac{4}{3}b_1\\0&-6&-12&b_3-7b_1\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&-1&-\frac{5}{3}b_1+\frac{2}{3}b_2\\0&1&2&-\frac{1}{3}b_2+\frac{4}{3}b_1\\0&0&0&b_1-2b_2+b_3\end{array}\right]$$

So our vectors span the plane described by $\boxed{b_1 - 2b_2 + b_3 = 0}$ (*) (since $\vec{b}$ satisfies this constraint equation $\Longleftrightarrow$ the system is consistent $\Longleftrightarrow$ $\vec{b} \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$).

Thinking of (*) as a "1×3 linear system", we can write solutions as

$$\begin{pmatrix}-b_3+2b_2\\b_2\\b_3\end{pmatrix} = b_2\begin{pmatrix}2\\1\\0\end{pmatrix} + b_3\begin{pmatrix}-1\\0\\1\end{pmatrix},$$

and notice that these two vectors span the plane all by themselves.

Definition: Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a list of vectors in $\mathbb{R}^m$. A linear dependence relation among these vectors is an equation
$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$
in which $c_1, \dots, c_n \in \mathbb{R}$ are not all 0.
A list of vectors with such a relation is said to be linearly dependent.

Ex 1 (cont'd) Let's show that $\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}$ is linearly dependent: we must find a nonzero solution to $\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Row-reducing the augmented matrix yields
$$\left[\begin{array}{ccc|c}1&2&3&0\\4&5&6&0\\7&8&9&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right] \rightsquigarrow \begin{cases}x_1 = x_3\\ x_2 = -2x_3\end{cases}$$
$$\Rightarrow \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \text{ works } \Rightarrow 1\begin{pmatrix}1\\4\\7\end{pmatrix} - 2\begin{pmatrix}2\\5\\8\end{pmatrix} + 1\begin{pmatrix}3\\6\\9\end{pmatrix} = \vec{0}$$
$x_3 = 1$ (say)

$\Rightarrow$ dependent. //

Ex 2 The vectors $\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \hat{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ have span all of $\mathbb{R}^3$, and clearly also have no linear dependence relation. What should we call such a set?

Definition: A list $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$ is linearly independent if it is not dependent: that is, if the only solution to
$$x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$$
is $\vec{x} = \vec{0}$. To check independence of $\vec{v}_1, \dots, \vec{v}_n$, you must prove the implication
$$\left.\begin{array}{c}c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}\\ (c_1, \dots, c_n \in \mathbb{R})\end{array}\right\} \Longrightarrow c_1 = \cdots = c_n = 0.$$

Suppose $c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 = \vec{0}$. Then $\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, hence $c_1 = c_2 = c_3 = 0$.

So $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are independent. Since they also span $\mathbb{R}^3$, they will be called a basis of $\mathbb{R}^3$. //

Here are 2 equivalent conditions on the columns $\vec{a}_1, \dots, \vec{a}_n$ of an $m \times n$ matrix A:

(I) The columns of A are LI

(II) They are all pivot columns
(i.e. every column of rref(A) contains a 'leading 1').

Check:

(II) $\Rightarrow$ (I): Suppose $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$.
Since (II) holds, row-reducing $[A \mid \vec{0}]$ yields

$$\left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & & & \vdots \end{array}\right]$$

(might have rows of 0's down here)

$\Rightarrow$ only solution is $x_1 = x_2 = \dots = x_n = 0$.

$\neg$(II) $\Rightarrow$ $\neg$(I): If $\mathrm{rref}[A \mid \vec{0}] = \left[\begin{array}{cccccc|c} 1 & \ddots & & * & & & 0 \\ & & 1 & * & & & 0 \\ & & & & 1 & & 0 \\ & & & & & \ddots\ 1 & \vdots \\ & & & & & & 0 \end{array}\right]$

has a non-pivot (say, i^{th}) column,
then there is a solution to $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$ in which (the free variable) x_i can be anything we want (in particular, nonzero). This gives a linear dependence relation. $\square$

EXERCISE Are the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

independent in $\mathbb{R}^4$? If not, find a linear dependence relation.

(2) If $m=n$, then (square matrix)

columns are independent $\iff \operatorname{rref}(A) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$\boxed{\iff}$ columns span $\mathbb{R}^m$.

This follows from Lecture 8, since the $m \times m$ identity matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ has no rows of "all zeroes" at the bottom.

(3) If a list $\vec{v}_1, \dots, \vec{v}_n$ contains $\vec{0}$, then it is dependent.

(Say $\vec{v}_1 = \vec{0}$. Then $1\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n = \vec{0}$ is a linear dependency.)

(4) A list is linearly dependent $\iff$ at least one $\vec{v}_i$ is a linear combination of the others

(If $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$, with some $c_k \neq 0$, then by rescaling we can assume $c_k = 1$. Then $\vec{v}_k = -c_1\vec{v}_1 - \cdots - c_{k-1}\vec{v}_{k-1} - c_{k+1}\vec{v}_{k+1} - \cdots - c_n\vec{v}_n$. Converse is also clear from this.)

EXERCISE Are the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix}$$

independent in $\mathbb{R}^4$? If not, find a linear dependence relation.

[Row reduce $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & \boxed{1} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Only 2 pivots! $\Rightarrow$ dependent. Can take $x_3 = 3$, $x_4 = 0 \rightsquigarrow x_1 = 1$, $x_2 = -2$; Check:

$$\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \\ -5 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ -4 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$]

Notice that: (A $m \times n$ matrix)

(1) If $m < n$, then the columns of A cannot be independent. (more than m vectors in $\mathbb{R}^m$)

(because there are only m rows to have leading 1's, and there are more columns than that)

§2. Bases

The first example involved 3 vectors spanning a plane thru $\vec{0}$ in $\mathbb{R}^3$; these turned out to be dependent, and we could span it with 2 independent vectors. The latter set is a basis:

Definition: Let $V \subset \mathbb{R}^n$ be a subspace; $\vec{v}_1, \dots, \vec{v}_k \in V$ are a basis for V if

- they are independent
- they span V

Given a basis, any vector $\vec{v} \in V$ can be expressed as a linear combination,

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k,$$

in one and only one way:

↖ b/c the $\{\vec{v}_i\}$ span V

If also $\vec{v} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$, then

$$0 = (d_1 - c_1)\vec{v}_1 + \cdots + (d_k - c_k)\vec{v}_k$$

$$\underset{\{\vec{v}_i\}\text{ independent}}{\Longrightarrow} d_i - c_i = 0 \ (\forall i) \Rightarrow d_i = c_i \ (\forall i).$$

$\boxed{\text{Ex 1}}$ $\hat{e}_1, \dots, \hat{e}_n$ is a basis of $\mathbb{R}^n$. //

Lemma: If $V \subset \mathbb{R}^n$ is a subspace with basis $\vec{v}_1, \dots, \vec{v}_k$, and let $\vec{w}_1, \dots, \vec{w}_l \in V$. If $l > k$, then $\vec{w}_1, \dots, \vec{w}_l$ are dependent.

Proof: Since $V = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$, we may express each $\vec{w}_j = \sum_{i=1}^{k} a_{ij}\vec{v}_i$, which yields a $k \times l$ matrix A. Since $k < l$, not all columns are pivot columns and so $A\vec{x} = \vec{0}$ has a nonzero solution $\vec{c}$. So

(since $A\vec{c} = \vec{0}$)

$$\sum_{j=1}^{l} c_j\vec{w}_j = \sum_{j=1}^{l} c_j \sum_{i=1}^{k} a_{ij}\vec{v}_i = \sum_{i=1}^{k} \underbrace{\Big(\cancel{\sum_{j=1}^{l} a_{ij}c_j}^{\,0}\Big)\vec{v}_i}_{\vec{0}}$$

gives a dependency. □

Theorem: Every nonzero subspace V has a basis. Any 2 bases for V have the same number of vectors.

Proof: Pick $\vec{v}_1 \neq \vec{0}$ in V. If this doesn't span V, pick $\vec{v}_2 \in V \setminus \text{span}\{\vec{v}_1\}$. If those don't span V, pick $\vec{v}_3$ in $V \setminus \text{span}\{\vec{v}_1, \vec{v}_2\}$; etc. At each stage the vectors are independent (why?). Since more than n vectors in $\mathbb{R}^n$ are dependent (Lemma + Ex. 1), the process must terminate.

If $\vec{v}_1, \dots, \vec{v}_k$ and $\vec{w}_1, \dots, \vec{w}_\ell$ are bases, then by the Lemma we can't have $\ell > k$. Swapping roles of "v" & "w", we can't have $k > \ell$ either. $\square$

Definition: The number of vectors in a basis of V is called its dimension, written $\dim(V)$.

Corollary: If $\dim(V) = k$, then any k vectors that span V are LI ($\Rightarrow$ basis) and any k LI vectors in V span V ($\Rightarrow$ basis).

The point is that multiplication by $\mathcal{E}$ (or $\mathcal{E}^{-1}$) is linear, so any relations on the columns of $\mathrm{rref}(A)$ are true on the columns of A. Since the Proposition is obviously true for $\mathrm{rref}(A)$

$$\begin{pmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 & * & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & * \cdots * & 0 & * \; \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * \; \cdots \\ & & & \vdots & & & & \end{pmatrix}$$

it is true for A. □

WARNING: Columns of A & $\mathrm{rref}(A)$ span different spaces.

Ex 2 Find a basis for the span of $\begin{pmatrix}1\\5\\9\end{pmatrix}, \begin{pmatrix}2\\6\\10\end{pmatrix}, \begin{pmatrix}3\\7\\11\end{pmatrix}, \begin{pmatrix}4\\8\\12\end{pmatrix}$ in $\mathbb{R}^3$.

pivot columns in rref(A)

Row-reduction gives $\begin{pmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{pmatrix} \to \begin{pmatrix}\boxed{1}&0&-1&-2\\0&\boxed{1}&2&3\\0&0&0&0\end{pmatrix}$

pivot columns of A

$\Rightarrow \begin{pmatrix}1\\5\\9\end{pmatrix}, \begin{pmatrix}2\\6\\10\end{pmatrix}$ are a basis of this span. //

Corollary: If $\dim(V) = k$, then any k vectors that span V are LI ($\Rightarrow$ basis) and any k LI vectors in V span V ($\Rightarrow$ basis).

Proof: If $\vec{v}_1, \dots, \vec{v}_k$ are LI but don't span V, adding $\vec{v}_{k+1} \in V \setminus \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ gives $k+1$ LI vectors in V.

If $\vec{v}_1, \dots, \vec{v}_k$ span but aren't LI, you can remove vector(s) without affecting the span, which leads to a basis with fewer than k vectors. □

Proposition: A basis for the span of the columns of a matrix is given by the pivot columns.

Proof: This is true because row-reduction is accomplished by multiplying by an invertible matrix (product of elementary matrices), $\mathrm{rref}(A) = \mathcal{E} \cdot A$.

§3. Inverses

Recall that for an $n\times n$ matrix A, an inverse is a matrix A^{-1} satisfying

$$A^{-1}A = I_n = AA^{-1}, \quad \text{where } I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

For $n=2$, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonzero determinant $\Delta := ad-bc$, recall that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and so $\dfrac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$.

For larger n, while there is a formula, it is computationally costly and a better approach is to use row-reduction.

Given $A = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{a}_1 & \cdots & \vec{a}_n \\ \downarrow & & \downarrow \end{pmatrix}$, we want

$$B = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix} \text{ with } AB = I_n, \text{ i.e.}$$

$$\begin{pmatrix} \uparrow & & \uparrow \\ A\vec{b}_1 & \cdots & A\vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \hat{e}_1 & \cdots & \hat{e}_n \\ \downarrow & & \downarrow \end{pmatrix},$$

which is equivalent to solving n systems

$(*)\quad A\vec{b}_1 = \hat{e}_1, \quad \ldots, \quad A\vec{b}_n = \hat{e}_n$

for $\vec{b}_1, \ldots, \vec{b}_n$. Since

$$A\vec{b}_k = A\begin{pmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{pmatrix} = b_{1k}\vec{a}_1 + \cdots + b_{nk}\vec{a}_n,$$

we must have that

$\quad \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ contains $\hat{e}_1, \ldots, \hat{e}_n$

$\Rightarrow \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_n\} = \mathbb{R}^n$

$\Rightarrow \operatorname{rref}(A)$ has a leading 1 in each row

$\Rightarrow \operatorname{rref}(A) = I_n$.

Conversely, if $\operatorname{rref}(A) = I_n$, then (for each k)

$$\operatorname{rref}[A \mid \hat{e}_k] = \left[I_n \;\middle|\; \begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix} \right]$$

and then $\begin{cases} x_1 = c_1 \\ \vdots \\ x_n = c_n \end{cases}$ solves the system $A\vec{x} = \hat{e}_k$

— that is, $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is the desired $\vec{b}_k$.

$$\text{rref}\,[A \mid I_n] = [I_n \mid \,?\,]$$

and the "?" will give you the desired B (with $AB = I_n$).

$\boxed{\text{EXERCISE}}$ Do this for $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & & \\ -2 & 2 & 1 & & 1 & \\ 0 & 1 & -1 & & & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & & & 3 & 1 & 1 \\ & 1 & & 2 & 1 & 1 \\ & & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} = B$$

... but how do I know that $BA = I_n$?

This is where the elementary matrices come in: recall that the result of any sequence of EROs on a matrix M takes the form $E_N \cdots E_1 M$, where the E_k are elementary matrices

replace $\begin{pmatrix} 1 & & \\ & \ddots & \\ a & & 1 \end{pmatrix}$ swap $\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ scale $\begin{pmatrix} 1 & & & \\ & \alpha & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

These have easy inverses

$\begin{pmatrix} 1 & & \\ & \ddots & \\ -a & & 1 \end{pmatrix}$ $\begin{pmatrix} \text{itself} \end{pmatrix}$ $\begin{pmatrix} 1 & & & \\ & 1/\alpha & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

and so $\mathcal{E} = E_N \cdots E_1$ has inverse $\mathcal{E}^{-1} = E_1^{-1} \cdots E_N^{-1}$.

Hence, if $\text{rref}(A) = I_n$, then

$$\mathcal{E} A = I_n \qquad (\mathcal{E} = \text{some product of elementary matrices})$$

and

$$\text{rref}\,[A \mid I_n] = \mathcal{E}[A \mid I_n] = [\mathcal{E}A \mid \mathcal{E}I_n] = [I_n \mid \mathcal{E}]$$

$$\Rightarrow \mathcal{E} = B \Rightarrow BA = I_n.$$

Conversely, if $CA = I_n$, then

$$A\vec{x} = \vec{0} \Rightarrow \vec{x} = CA\vec{x} = C\vec{0} = \vec{0}$$

says that $A\vec{x} = \vec{0}$ has only the trivial solution; so A has no non-pivot columns.

Hence $\text{rref}(A) = I_n$.

Moreover, if $CA = I_n = AB$, then

$$B = I_n B = CAB = CI_n = C.$$

... so all this proves what, exactly?

Thus:

Theorem: An $n \times n$ matrix A is invertible if (and only if) one of the equivalent statements

$$\begin{cases} \text{(i)} & \operatorname{rref}(A) = I_n \\ \text{(ii)} & \exists B \text{ with } AB = I_n \\ \text{(iii)} & \exists C \text{ with } CA = I_n \end{cases}$$

holds. The inverse of A is then $A^{-1} = B = C$.

$\boxed{\text{Ex 1}}$ Is $\begin{pmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ invertible?

$\boxed{\text{Ex 2}}$ What about $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?

and so $\mathcal{E} = E_N \cdots E_1$ has inverse $\mathcal{E}^{-1} = E_1^{-1} \cdots E_N^{-1}$.

Hence, if $\operatorname{rref}(A) = I_n$, then

$$\mathcal{E} A = I_n \qquad (\mathcal{E} = \text{some product of elementary matrices})$$

and

$$\operatorname{rref}[A \mid I_n] = \mathcal{E}[A \mid I_n] = [\mathcal{E}A \mid \mathcal{E}I_n] = [I_n \mid \mathcal{E}]$$

$$\Rightarrow \mathcal{E} = B \Rightarrow BA = I_n.$$

Conversely, if $CA = I_n$, then

$$A\vec{x} = \vec{0} \Rightarrow \vec{x} = CA\vec{x} = C\vec{0} = \vec{0}$$

says that $A\vec{x} = \vec{0}$ has only the trivial solution; so A has no non-pivot columns.

Hence $\operatorname{rref}(A) = I_n$.

Moreover, if $CA = I_n = AB$, then

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... so all this proves what, exactly?

This:

Theorem: An $n \times n$ matrix A is invertible if (and only if) one of the equivalent statements

$$\begin{cases} \text{(i) } \operatorname{rref}(A) = I_n \\ \text{(ii) } \exists\, B \text{ with } AB = I_n \\ \text{(iii) } \exists\, C \text{ with } CA = I_n \end{cases}$$

holds. The inverse of A is then $A^{-1} = B = C$.

$\boxed{\text{Ex 1}}$ Is $\begin{pmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ invertible?

Yes. No calculation required! You can see that it has 4 pivots $\Rightarrow \operatorname{rref}(A) = I_4$.

$\boxed{\text{Ex 2}}$ What about $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$?

No. You can see at once that it will have a row of all zeroes in its RREF.

Side remark: A^{-1} can be used to solve $A\vec{x} = \vec{b}$ — multiplying by it gives $\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}$.

Here are some more conditions that are equivalent to the ones in the Theorem:

- $\operatorname{rank}(A) = n$ (b/c I_n has n pivots)
  [a.k.a. "A is nonsingular"]
- $A\vec{x} = \vec{0}$ has only the trivial solution
  (b/c this is equivalent to all columns having a leading 1)
- $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$
  (b/c this is equivalent to all rows having a leading 1)

This:

Theorem: An $n\times n$ matrix A is invertible if (and only if) one of the equivalent statements

$$\begin{cases} \text{(i) } \operatorname{rref}(A) = I_n \\ \text{(ii) } \exists\, B \text{ with } AB = I_n \\ \text{(iii) } \exists\, C \text{ with } CA = I_n \end{cases}$$

holds. The inverse of A is then $A^{-1} = B = C$.

Define T to be invertible if there exists a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ with $S(T(\vec{x})) = \vec{x} = T(S(\vec{x}))$ $(\forall \vec{x})$, and write $T^{-1} = S$.

Since matrix multiplication matches composition of transformations, it is clear that A is invertible if and only if

- T is invertible

and then the matrix of T^{-1} is A^{-1}.

So things like rotations are represented by invertible matrices.

Here are some more conditions that are equivalent to the ones in the Theorem:

- $\operatorname{rank}(A) = n$ (b/c I_n has n pivots)
  [a.k.a. "A is nonsingular"]
- $A\vec{x} = \vec{0}$ has only the trivial solution
  (b/c this is equivalent to all columns having a leading 1)
- $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$
  (b/c this is equivalent to all rows having a leading 1)

The corresponding statements for linear transformations $T: \mathbb{R}^n \to \mathbb{R}^n$ (with matrix A) are

- T is 1-to-1
- T is onto

Since 1-to-1 and onto are different concepts, it may seem strange that these are equivalent statements here. Again, that's b/c the domain & codomain are both $\mathbb{R}^n$ (corresponding to the fact that A is $n\times n$).