Math 3300

Final Exam Review

April 24, 2026

Format

- This will be a 2-hour exam, with roughly 15 multiple choice problems and 5 pages of free response questions.
- The exam is on Monday May 4 in McMillin G052, from 8:30-10:30 **PM**.
- Bring only your ID and pen/pencils.
- As before, I will be grading this on Gradescope, so write your answer in the space provided (or tell me which extra page to look at).
- Material since Exam 2 is emphasized; overall, I'd suggest studying Chapters 5 and 6 and sections 7.1-2.
- Main themes include: diagonalization/diagonalizability; symmetric and orthogonal matrices; applications to dynamical systems, least-squares/data fitting, and quadratic forms.
- I'll discuss an extra credit opportunity at the end.

How to review

- For a review of the earlier material, please consult the study guides for Exams 1 and 2, and the exams themselves.
- While you did not have to use what you had learned about writing matrices of linear transformations with respect to a basis or solving dynamical systems on those exams; you may have to do some of that here.
- You may also want to review how to compute rank of a matrix, & how to parametrize solutions of a linear system.
- Besides this study guide (posted on the webpage), **there will be practice problems** available Monday on Canvas under Files.
- I will have office hours specifically for Math 3300 on Tuesday and Thursday next week from 3-4 PM.
- What follows is a review of the newer material.

Eigenstuff

When is an $n \times n$ matrix diagonalizable?

Answer 1: when the sum of $\dim(E_{\lambda_i})$'s is equal to n, so that we get an eigenbasis $\mathcal{B}$, and $A = P_{\mathcal{B}} D P_{\mathcal{B}}^{-1}$.
Answer 2: Equivalently, when each $\dim(E_{\lambda_i})$ = multiplicity of λ_i as a root of $\det(A - \lambda I_n)$.

You should know how to diagonalize (if possible) when these dimensions are not all 1 — this requires writing a basis for each eigenspace.

Complex eigenvalues and eigenvectors of real matrices: know that in the 2×2 case,

$$A = P \begin{pmatrix} a & -b \\ b & a \end{pmatrix} P^{-1},$$

how to compute this. More generally, know that these come in conjugate pairs.

Inner products and orthogonality

- Properties of the dot product, general inner product
  - axioms: bilinearity, symmetry, positive-definiteness; how to check
  - triangle inequality, Cauchy-Schwarz, Pythagorean theorem
  - distance and angle in $\mathbb{R}^n$
- Orthogonality
  - $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$ (or $\langle \vec{u}, \vec{v} \rangle = 0$)
  - orthogonal sets, relation to linear independence
  - orthogonal complements, orthogonal and orthonormal bases

Inner products and orthogonality

- Orthogonal projections
  - given an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_k$ for $W \subset V$, $\text{proj}_W \vec{y} = \sum_{i=1}^{k} \langle \vec{y}, \vec{u}_i \rangle \vec{u}_i$
  - given an orthogonal basis $\vec{v}_1, \ldots, \vec{v}_k$ for $W \subset V$, $\text{proj}_W \vec{y} = \sum_{i=1}^{k} \frac{\langle \vec{y}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle} \vec{v}_i$
  - orthogonal ("Fourier") decomposition of a vector: e.g. if $\vec{u}_1, \ldots, \vec{u}_n$ is an o.n. basis of V, $\vec{y} = \sum_{i=1}^{n} \langle \vec{y}, \vec{u}_i \rangle \vec{u}_i$
  - geometric interpretation of orthogonal projection (distance-minimizing), relation to least-squares solutions
- Orthogonal matrices
  - $U^{\mathsf{T}} = U^{-1}$, or equiv.: U is **square** with o.n. columns
  - are invertible
  - when doing Gram-Schmidt for orthogonal diagonalization, do "full G-S" (since you need ortho*normal* columns for P)
  - can also write the above condition as $U^{\mathsf{T}} U = I$ (what does this mean for the determinant?)

Inner products and orthogonality

- more projection formulas
  - given a **non-square** matrix U with o.n. columns $\{\vec{u}_i\}$,
    $$UU^{\mathsf{T}}\vec{y} = U\begin{pmatrix}\vec{u}_1\cdot\vec{y}\\ \vdots \\ \vec{u}_k\cdot\vec{y}\end{pmatrix} = \textstyle\sum_{i=1}^{k}(\vec{u}_i\cdot\vec{y})\vec{u}_i = \mathrm{proj}_{\mathrm{Col}(U)}\vec{y}$$
    So UU^{T} is the matrix of $\mathrm{proj}_{\mathrm{Col}(U)}$.
  - useful together with Gram-Schmidt (use G-S to get an o.n. basis $\{\vec{u}_i\}$ of a subspace W)
- Gram-Schmidt (for dot or more general inner products)
  - given a basis $\vec{w}_1, \dots, \vec{w}_k$ for a subspace $W \subset \mathbb{R}^n$ (e.g. an eigenspace), G-S gives a formula for an orthogonal basis:
    $$\vec{v}_1 = \vec{w}_1, \;\; \vec{v}_2 = \vec{w}_2 - \tfrac{\vec{w}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1, \;\; \text{etc.}$$
  - "full G-S" is when you then take $\vec{u}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|} = \frac{\vec{v}_i}{\sqrt{\vec{v}_i\cdot\vec{v}_i}}$ to get an orthonormal basis. You need to do this to construct the matrices U described above.

Symmetric matrices

These are square matrices A with $A = A^{\mathsf{T}}$. (Not to be confused with orthogonal matrices, whose transpose gives their *inverse*.)

Spectral Theorem

If A is symmetric, then $A = PDP^{\mathsf{T}}$ for some orthogonal matrix P and diagonal matrix D. (It is *orthogonally diagonalizable*, which is even better than just being diagonalizable.)

You should be able to perform orthogonal diagonalization.

Symmetric matrices ($A = A^{\mathsf{T}}$)

Spectral Theorem

If A is symmetric, then $A = PDP^{\mathsf{T}}$ for some orthogonal matrix P and diagonal matrix D.

You should also understand why

- $\langle \vec{x}, \vec{y} \rangle := \vec{x}^{\mathsf{T}} A \vec{y} = \vec{x} \cdot A\vec{y}$ is symmetric
- eigenvalues of A are real
- eigenvectors with distinct eigenvalues are orthogonal

and **be able to reproduce these proofs** (easier than you think: see pp. 1-2 of Lecture 36).

Know how to write down the matrix of a quadratic form $Q(\vec{x}) = \vec{x}^{\mathsf{T}} A \vec{x}$. Also, how to determine the principal axes for $Q(\vec{x})$ and eliminate cross-terms, decide whether Q is positive-definite, etc.

Applications

- discrete dynamical systems
  - for 2×2, be prepared to do "complex" systems
  - be able to recognize attractor/repeller/saddle point
- continuous dynamical systems
  - systems of linear differential equations $\frac{d\vec{x}}{dt} = A\vec{x}$, with $\vec{x}(0)$ given
  - only worry about the real eigenvalues case
- data-fitting/linear modeling
  - how to convert to least-squares problems
- least-squares solutions to $A\vec{x} = \vec{b}$
  - solve $A\hat{x} = \hat{b}$, where $\hat{b} := \mathrm{proj}_{\mathrm{Col}(A)}\vec{b}$
  - normal equations $A^{\mathsf{T}}A\hat{x} = A^{\mathsf{T}}\vec{b}$
  - when unique?

- describing geometry of solution set to $Q(\vec{x}) = 1$

The following (for 2×2) **will be useful on the exam**:

Example

What is the geometry of $1 = Q(\vec{x}) = ax_1^2 + bx_1x_2 + cx_2^2$? Let $A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$ be the matrix of Q. We know that an orthogonal change of coordinates makes this $1 = \lambda_1 y_1^2 + \lambda_2 y_2^2$. Now

$$\lambda_1 \lambda_2 = \det(A) = ac - \frac{b^2}{4},$$

while

$$\lambda_1 + \lambda_2 = \mathrm{tr}(A) = a + c.$$

So if $\det(A) > 0$ and $\mathrm{tr}(A) > 0$, then Q is positive definite ($\lambda_1, \lambda_2 > 0$) and $Q(\vec{x}) = 1$ defines an **ellipse**. If $\det(A) < 0$, then (regardless of $\mathrm{tr}(A)$) Q is indefinite (A has one positive, one negative eigenvalue), and $Q(\vec{x}) = 1$ defines a **hyperbola**.

Survey and Evaluation

- In Canvas under Assignments, you will see "Spring 2026 Math 3300 End of Semester Survey".
- It opens at 2 PM today (right after class) and closes on May 4 (right before the exam).
- While it says "1 pts", completing it is worth 2 points of extra credit on your Final Exam (which is out of 40).
- The survey is targeted at getting feedback on specific aspects of the course which you felt contributed the most to your learning and to the classroom environment.
- It is administered by Dr. Ali York, an evaluations specialist in the Institutional Effectiveness office on campus. She is the only one who will see your responses.
- Please also complete the (separate) WashU Course Evaluation for this class.