

Lecture 1 : Linear Systems

Geometric viewpoint

For simplicity, start with systems of n linear equations in n unknowns, where $n=2$ or 3 .

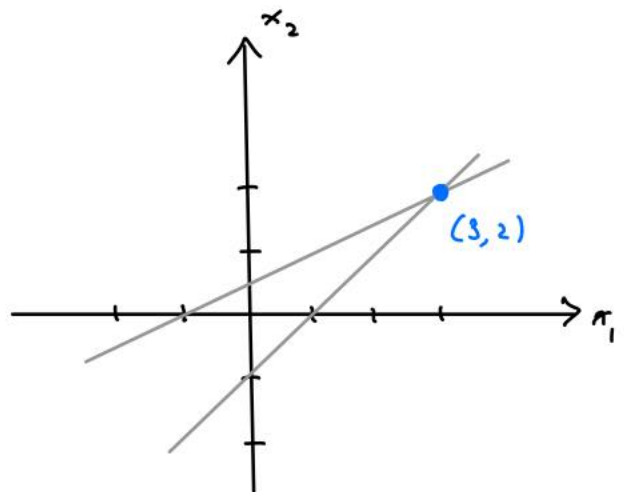
These will have 3 presentations :

- (a) by "row" equations
- (b) by "column" equations
- (c) by "matrix" equation.

Ex 1 / Consider the system (in form (a))

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + x_2 = -1 \end{cases},$$

with accompanying picture :



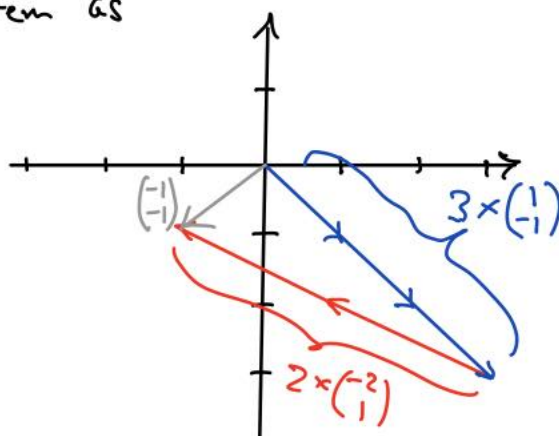
... from which we see that $(x_1, x_2) = (3, 2)$ is the unique solution of the system.

(b) Now we can rewrite the system as

$$x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

which asks the question:

"Can we produce $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ as a
linear combination of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$?"



The answer, as shown in the picture, is YES.

(c) The matrix form of the equation is

$$\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

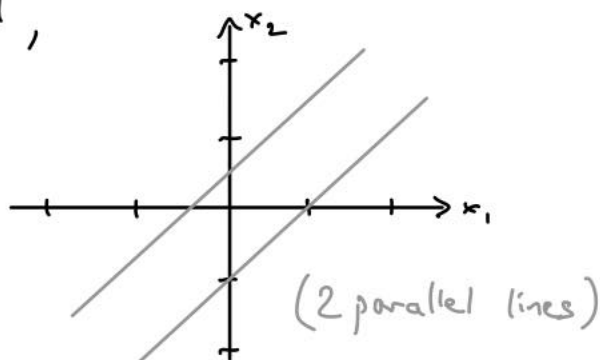
" $A \cdot \vec{x} = \vec{b}$ "

The system in the Example is called consistent
because a solution exists. Here is an inconsistent one:

Ex 2/ If we change the first equation in Ex. 1 to

$$2x_1 - 2x_2 = -1,$$

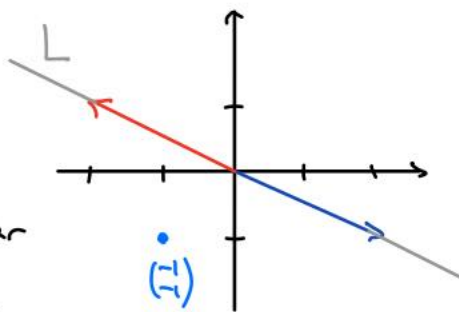
then picture (a) becomes



While in (b) we have

$$x_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

which is impossible (as any linear combination of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ lies on the line L , and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ does not).



Ex 3 / Finally, if we change $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ to something lying on the line L , say $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$, then (in (b)) there are many linear combinations of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ that will do. Correspondingly, the two parallel lines in picture (a) of Example 2 merge, and we have infinitely many solutions ($\Rightarrow$ consistent).

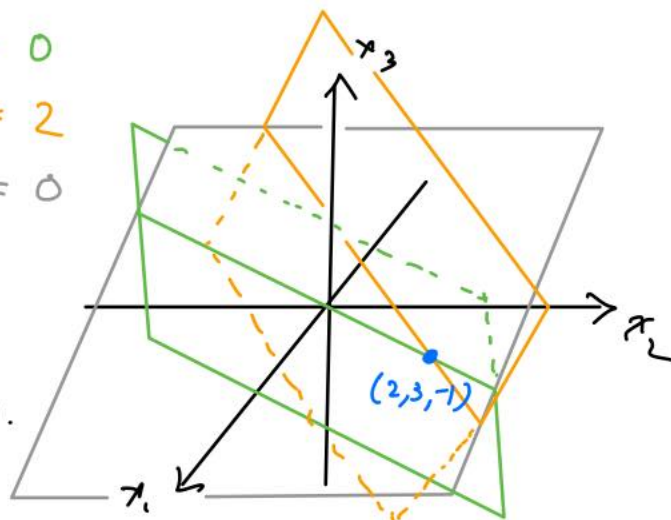
Turning to $n=3$, here is a consistent example:

Ex 4 / (a) Row

$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ -x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_3 = 0 \end{cases}$$

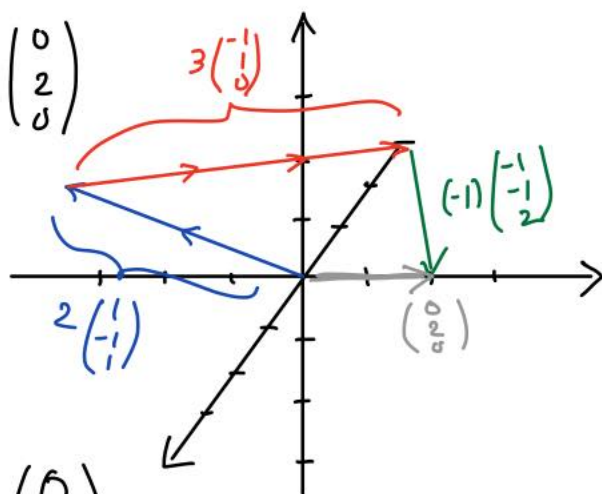
3 planes in space

Note that two of the planes pass through the origin $(0,0,0)$.
(why?)



COLUMN

$$(b) \quad x_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

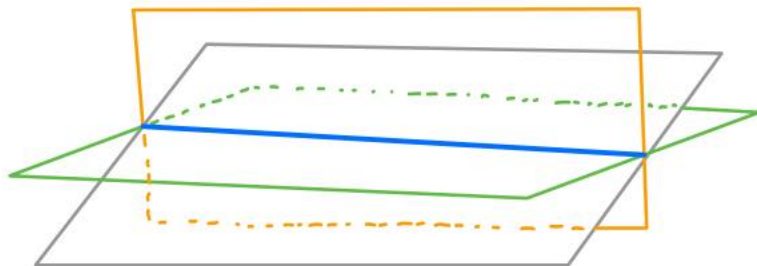


MATRIX

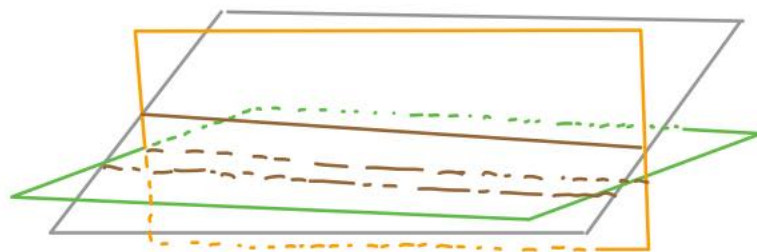
$$(c) \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Ex 5 / If in Ex. 4(a), we made the last equation $x_3 = -1$,

we would get a line as solution set :



On the other hand, if we move the $x_3 = -1$ plane up or down to $x_3 = a$ ($\neq -1$), then we get the picture



so that there are no common solutions (and the system is inconsistent).

Correspondingly, what happens with the vector equations?

The linear combinations

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \text{ include } \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \text{ but NOT } \begin{pmatrix} 0 \\ 2 \\ a \end{pmatrix} \text{ for } a \neq -1.$$

From these examples we can (informally) glean that

(i) There are 3 possibilities for linear systems:

no solutions, one solution, or infinitely many.

(ii) The n equations have a common solution (i.e.

are consistent) $\iff$ the column vector $\vec{b}$ on the right-hand-side of the vector equation can be written as a linear combination of the column vectors on the left-hand-side.

(iii) The equations have a common solution for every $\vec{b}$

$\iff$ linear combinations of the left-hand-side column vectors fill up all of n -space.

Algebraic viewpoint

This will be a first glimpse of Gaussian elimination / row operations, to be made more systematic in subsequent lectures.

Ex 6 / Find the solution set (possibly empty) of the system

$$\left. \begin{array}{l} (p_1) \quad x_1 + x_2 + x_3 = 9 \\ (p_2) \quad 2x_1 + 4x_2 - 3x_3 = 1 \\ (p_3) \quad 3x_1 + 6x_2 - 5x_3 = 0 \end{array} \right\}$$

eliminate x_1 in last 2 equations $\downarrow$ $p_2 \mapsto p_2 - 2p_1$
 $p_3 \mapsto p_3 - 3p_1$

$$\left. \begin{array}{l} (p_1) \quad x_1 + x_2 + x_3 = 9 \\ \text{new } (p_2) \quad 2x_2 - 5x_3 = -17 \\ \text{new } (p_3) \quad 3x_2 - 8x_3 = -27 \end{array} \right\}$$

eliminate x_2 in last equation $\downarrow$ $p_3 \mapsto p_3 - \frac{3}{2}p_2$

$$\left. \begin{array}{l} (p_1) \quad x_1 + x_2 + x_3 = 9 \\ (p_2) \quad 2x_2 - 5x_3 = -17 \\ \text{new } (p_3) \quad -\frac{1}{2}x_3 = -\frac{3}{2} \end{array} \right\}$$

So $x_3 = 3$, and back-substituting in (p_2) gives $2x_2 - 15 = -17$
 $\Rightarrow x_2 = -1$, whereupon substituting in (p_1) gives $x_1 - 1 + 3 = 9$
 $\Rightarrow x_1 = 7$. You can check that $(7, -1, 3)$ solves the original system. //

Short-hand
 $\rightsquigarrow$

"AUGMENTED MATRIX"

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right]$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 3 & -8 & -27 \end{array} \right]$$

$$\downarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 2 & -5 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array} \right]$$

Why does this work? Applying

Elementary Row Operations

(a) Replace an equation/row by itself + multiples of other equations/rows

(b) Swap two equations/rows

(c) Scale an equation/row (multiply by a nonzero constant)

produces a new "row-equivalent" system of equations whose solution set certainly includes all the old solutions.

In fact, since (a)-(c) are reversible, the new solution set is the same:

Row-equivalent systems are equivalent.

Here's an example that uses all 3 operations:

Ex 7 /

$$\left. \begin{array}{l} x_3 - x_4 = -1 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 = 2 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 = 3 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 = 6 \end{array} \right\} \rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array} \right] \xleftarrow{(b): r_1 \leftrightarrow r_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array} \right] \xrightarrow{(c): r_i \mapsto \frac{1}{2}r_i} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{array} \right] \xrightarrow{(a): \begin{array}{l} r_3 \mapsto r_3 - 2r_1 \\ r_4 \mapsto r_4 - 3r_1 \end{array}}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{array} \right] \xrightarrow[\substack{\text{Col:} \\ p_3 \rightarrow p_3 - p_2 \\ p_4 \rightarrow p_4 - 3p_2}]{\text{Col:}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 6 \end{array} \right]$$

STOP. The 3rd line corresponds to the equation $0 = 2$,
and so the system is inconsistent (no solutions). //