

Lecture 11: More stuff about matrices

Recall the three equivalent statements about an $n \times n$ (square) matrix A :

$$(S1) \text{ rref}(A) = \mathbb{I}_n$$

$$(S2) \exists B \text{ with } AB = \mathbb{I}_n$$

$$(S3) \exists C \text{ with } CA = \mathbb{I}_n.$$

If these hold, A is invertible, with inverse $A^{-1} := B = C$. This inverse matrix can be computed by taking

$$\text{rref}[A \mid \mathbb{I}_n] = [\mathbb{I}_n \mid A^{-1}].$$

There is a long list of additional conditions equivalent to (S1)–(S3) in the book. Here are a few of them:

$$(S4) A\vec{x} = \vec{0} \text{ has only the trivial solution}$$

is equivalent to all columns having a leading 1.

Since A is square, this is the same as $\text{rref}(A) = \mathbb{I}_n$.

(S5) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$

is equivalent to all rows having a leading '1',
which likewise is the same as $\text{ref}(A) = I_n$ (because
 A is square).

The corresponding statements for the linear transformation
 $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by A are

(S4) T is 1-1

(S5) T is onto.

Since 1-1 and onto are different concepts, it may
seem strange that these are equivalent statements here.
Again, that's because the domain and codomain are both
 $\mathbb{R}^n$ (corresponding to the fact that A is $n \times n$).

Definition: A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$
is invertible if there exists a linear transformation
 $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$(*) \quad \begin{cases} S(T(\vec{x})) = \vec{x} \\ T(S(\vec{x})) = \vec{x} \end{cases} \quad \text{for every } \vec{x} \in \mathbb{R}^n.$$

We write $T^{-1} := S$ (it's clear from (*) that this is unique).

Theorem: T is invertible $\Leftrightarrow$ its matrix A is invertible, and then T^{-1} has matrix A^{-1} .

Proof: If T is invertible, it is 1-1 and onto (otherwise (*) can't hold). By our equivalent condition, A is invertible. Since composition of L.T.s corresponds to matrix multiplication, if $S(\vec{x}) := A^{-1} \cdot \vec{x}$ then $S(T(\vec{x})) = A^{-1} \cdot A \cdot \vec{x} = \mathbb{I}_n \cdot \vec{x} = \vec{x}$ and vice-versa $\Rightarrow S = T^{-1}$. $\square$

Our last equivalent condition, then, is

(S6) T is invertible.

Ex 1 / Is $A = \begin{pmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ invertible?

YES. No calculation required! You can see clearly that it has 4 pivots (hence will have $\text{ref}(A) = \mathbb{I}_4$).

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Ex 2/ Is $A = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ invertible?

NO. You can see at once that it will have a row of all zeroes in its RREF.

Ex 3/ Is rotation $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ about the origin (thru angle θ) invertible as a L.T.?

YES. $R_{-\theta}$ is the inverse. You don't need to calculate matrices here.

LU Factorization

We can apply what we've learned about elementary matrices & row-reduction to more general matrices A :

Suppose A can be reduced to a row-echelon matrix U entirely by replace operations that add multiples of rows to those below them. The elementary matrices which accomplish these operations are all of the form

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & a & \\ & & & 1 \end{pmatrix}$$

with the "a" below the diagonal. We have

$$E_N \cdots E_1 A = U$$

$$A = (E_N \cdots E_1)^{-1} U = \underbrace{(E_1^{-1} \cdots E_N^{-1})}_{=: L} U$$

Here A need not be square
(U will have the same dimensions),
but L will always be.

= lower triangular
matrix with 1s
on the diagonal

Ex 4 / $\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix} = A$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \downarrow \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 6 & -4 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \downarrow \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 10 & 4 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \downarrow \begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = U$$

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$$

So

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix},$$

$$\text{and } A = L \cdot U.$$

Why is this useful? Suppose that you want to solve $A\vec{x} = \vec{b}$ for lots of $\vec{b}$'s.

Approach 1: $\vec{x} = A^{-1}\vec{b}$ (works if A is invertible)

Approach 2: $L(U\vec{x}) = \vec{b}$ $\rightsquigarrow$ $\begin{cases} L\vec{y} = \vec{b} \\ U\vec{x} = \vec{y} \end{cases}$

This is more efficient & general.

Ex 5 / A as above, $\vec{b} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$. For $L\vec{y} = \vec{b}$,

$$\text{use } \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{array} \right], \text{ get } \begin{cases} y_1 = -7 \\ y_2 = 5 + y_1 = -2 \\ y_3 = 2 - 2y_1 + 5y_2 = 2 + 14 - 10 = 6 \end{cases}$$

For $U\vec{x} = \vec{y}$, use

$$\left[\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array} \right] \rightsquigarrow \begin{cases} x_3 = -6 \\ -2x_2 = x_3 - 2 = -8 \Rightarrow x_2 = 4 \\ 3x_1 = 7x_2 + 2x_3 - 7 = 9 \\ \Rightarrow x_1 = 3 \end{cases}$$

$$\text{So } \vec{x} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}.$$

You can ignore the application in this section.

Partitioned matrices (matrix multiplication in "blocks")

Say we have a 4×4 matrix A and a 4×2 matrix B .

$$\begin{array}{c}
 \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array} \right) \cdot \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \hline b_{31} & b_{32} \\ b_{41} & b_{42} \end{array} \right) = \left(\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \hline c_{31} & c_{32} \\ c_{41} & c_{42} \end{array} \right) \\
 A \qquad \qquad \qquad B \qquad \qquad \qquad C
 \end{array}$$

We could do the traditional row-by-column multiplication, which gives

$$c_{ij} = \sum_{k=1}^4 a_{ik} b_{kj}.$$

It can be more convenient for some purposes to write

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \cdot \left(\begin{array}{c} B_1 \\ \hline B_2 \end{array} \right) = \left(\begin{array}{c} C_1 \\ \hline C_2 \end{array} \right)$$

so e.g. $C_1 = A_{11} \cdot B_1 + A_{12} \cdot B_2$,

$\underbrace{\hspace{1.5cm}}_{\text{matrix products}}$

and c_{11} is obtained by the row-column multiplications shown:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}.$$

Your book has more examples & pictures of this.

Skip the business on "column-row" product expansions.

Ex 6 / Find formulas for X, Y, Z in terms of A, B, C , if

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \left[\begin{array}{c|c} I & 0 \\ \hline X & Y \end{array} \right] = \left[\begin{array}{c|c} 0 & I \\ \hline Z & 0 \end{array} \right]$$

$$\left[\begin{array}{c|c} A+BX & BY \\ \hline C & 0 \end{array} \right]$$

$$\Rightarrow \begin{cases} A+BX = 0 & \Rightarrow BX = -A \\ C = Z \\ BY = I \end{cases}$$

$$\Rightarrow \begin{cases} Y = B^{-1} \\ Z = C \\ X = -B^{-1}A \end{cases}$$

