Lecture 12 : Determinants

Suppose you were looking for a function

$$\det : \left\{ \begin{array}{c} n \times n \ \text{real} \\ \text{matrices} \end{array} \right\} \longrightarrow \mathbb{R}$$

with the following 3 properties, where we think of an $n \times n$ matrix A as an ordered n-tuple of row vectors $\begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{pmatrix}$:

(i) **Multilinearity**: $\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ a\vec{r}_i + b\vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = a \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} + b \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}$

(ii) **Antisymmetry**: $\det \begin{pmatrix} \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = - \det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_i \\ \vdots \end{pmatrix}$

(iii) **Normalization**: $\det \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix} = 1 .$

As we'll see, not only does such a function exist, it is unique. Anticipating this uniqueness, we'll call this function <u>the determinant</u>, written $\det(A)$ or $|A|$.

<u>Theorem 1</u> : If 2 rows of A are equal, $\det A = 0$.

<u>Proof</u>: If $\vec{r}_i = \vec{r}_j$, then by (ii), $\det A = -\det A$
$$\Rightarrow 2 \det A = 0 \Rightarrow \det A = 0. \qquad \square$$

$\boxed{2\times 2 \text{ matrices}}$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the

defining properties (i)~(iii) to compute $\det A$:

__Case 1__: $a = c = 0$. $|A| = \begin{vmatrix} b\vec{e}_2 \\ d\vec{e}_2 \end{vmatrix} \underset{(i)}{=} bd \begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix} \underset{\text{Thm.1}}{=} 0$.

__Case 2__: $a = 0$. $|A| = \begin{vmatrix} b\vec{e}_2 \\ c\vec{e}_1 + d\vec{e}_2 \end{vmatrix} \underset{(i)}{=} bc \begin{vmatrix} \vec{e}_2 \\ \vec{e}_1 \end{vmatrix} + bd \begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix}^{\,0}$

$$= -bc \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \end{vmatrix} \underset{(iii)}{=} -bc.$$

__Case 3__: $a \neq 0$.

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \underbrace{\frac{c}{a}\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0} \underset{(i)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ c & \frac{bc}{a} \end{vmatrix}$

$\underset{(i)}{=} \begin{vmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{vmatrix} \underset{(i)}{=} \begin{vmatrix} a & 0 \\ 0 & d-\frac{bc}{a} \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d-\frac{bc}{a} \end{vmatrix}}_{0 \text{ by Case 1}} \underset{(i)}{=} a\left(d-\frac{bc}{a}\right)\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1 \text{ by (iii)}}$

$$= ad - bc.$$

In each case, we get $\underline{ad - bc}$.

__Theorem 2__: If A is an $n \times n$ upper (or lower) triangular

matrix, then $\det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

__Proof__: (will do for 3×3 matrices)

$|A| = \begin{vmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{vmatrix} = \begin{vmatrix} \alpha\vec{e}_1 + a\vec{e}_2 + b\vec{e}_3 \\ \beta\vec{e}_2 + c\vec{e}_3 \\ \gamma\vec{e}_3 \end{vmatrix} = \alpha\beta\gamma \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}^{\,1} + a c\gamma \begin{vmatrix} \vec{e}_1 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}^{\,0} + a\beta\gamma \begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}^{\,0}$

$\qquad\qquad + ac\gamma \begin{vmatrix} \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}^{\,0} + b\beta\gamma \begin{vmatrix} \vec{e}_3 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}^{\,0} + bc\gamma \begin{vmatrix} \vec{e}_3 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}^{\,0}$

$$= \alpha\beta\gamma. \qquad \square$$

$\boxed{3 \times 3 \text{ matrices}}$ Let $A = \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \\ d & e & f \end{pmatrix}$. We have by (i)

$$|A| = \alpha \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{vmatrix}. \quad (*)$$

Now notice that

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} - a\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ d & e & f \end{vmatrix}}_{=0} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ d & e & f \end{vmatrix} - d\begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix}$$

by using Thm 1 (and (i) repeatedly.

In this way, $(*)$ becomes

$$\alpha \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & 0 & c \\ d & 0 & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & 0 \\ d & e & 0 \end{vmatrix} \quad (**)$$

Applying the same steps as for $n=2$ to the bottom 2 rows in $(**)$ gives:

$$\alpha(bf-ec)\underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{|\mathbb{I}_3| = 1} + \beta(af-dc)\underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{\substack{=-|\mathbb{I}_3|=-1 \\ (ii)}} + \gamma(ae-bd)\underbrace{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{\substack{=-\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}=(-|\mathbb{I}_3|) \\ =+1}}$$

$$= \alpha \underbrace{\begin{vmatrix} b & c \\ e & f \end{vmatrix}}_{=: A_{\widehat{11}}} - \beta \underbrace{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}_{=: A_{\widehat{12}}} + \gamma \underbrace{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}_{=: A_{\widehat{13}}}, \text{ which is the } \underline{\text{Laplace expansion}} \text{ of } \det(A) \text{ along the first row.}$$

$\boxed{n \times n \text{ matrices}}$ Let $A_{\widehat{ij}} := (n-1) \times (n-1)$ matrix obtained from A by deleting the ith row & jth column.

<u>Theorem 3</u>: For any fixed i,

"Laplace expansion in the ith row"

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot \underbrace{(-1)^{i+j} \cdot \det(A_{\widehat{ij}})}_{=: C_{ij} = (i,j)\underline{\text{th}} \text{ cofactor}}.$$

$$Ex/ \quad \begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 3 \\ 0 & 2 & 0 \end{vmatrix} = (-1)^{3+2} \, 2 \, \begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = -2 \cdot 9 = -18. \qquad //$$
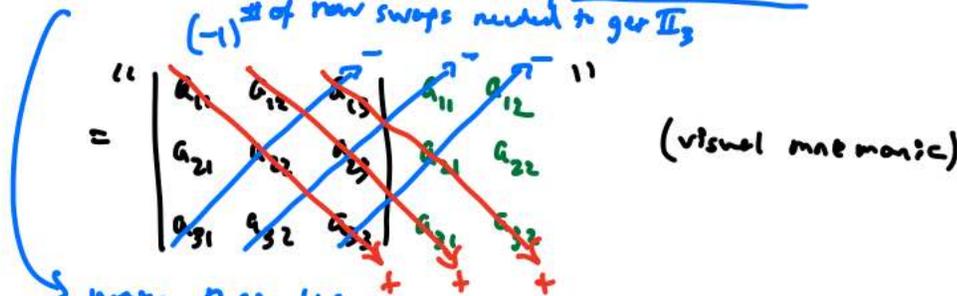
Another approach to 3×3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{3} a_{1i} \vec{e}_i \\ \sum_{j=1}^{3} a_{2j} \vec{e}_j \\ \sum_{k=1}^{3} a_{3k} \vec{e}_k \end{vmatrix} = \sum_{i,j,k=1}^{3} a_{1i} \, a_{2j} \, a_{3k} \begin{vmatrix} \vec{e}_i \\ \vec{e}_j \\ \vec{e}_k \end{vmatrix}$$

0 unless i,j,k all different!

$$= a_{11} a_{22} a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12} a_{23} a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13} a_{21} a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11} a_{23} a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12} a_{21} a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13} a_{22} a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}$$

(ii): the determinants of the permutation matrices shown are $(-1)^{\# \text{ of row swaps needed to get } \mathbb{I}_3}$

$$= \quad " \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} "$$

(visual mnemonic)

$\longmapsto$ notice that this is also the # of column swaps needed to reach $\mathbb{I}$, (you perform the same swaps in the reverse order).

$\Downarrow$

The same definition of determinant but w/ column vectors replacing row vectors, yields the same answer! So we get

Theorem 3: $\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det (A_{\widehat{ij}})$, for any fixed j;

and $\quad$ (Laplace expansion in the jth column)

Theorem 4 : $\det A = \det A^T$.

Ex /
$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -4 \end{array}$$

$$= 3 \left(\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \right) - 1 \left(\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right)$$

$$= 3 \cdot (-2) - 1 \cdot 2 = -8.$$

More on this next lecture.

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| An economic application : Leontief model |

Here is the general problem: Suppose the consumer sector of our economy demands, over the course of a year,

final demand vector $\vec{d} = \begin{pmatrix} 60 \\ 40 \\ 10 \end{pmatrix}$ units of manufacturing (MF)
units of transport (TR)
units of agriculture (AG),

Those sectors must actually produce _more_ than this, since the aggies need their Deeres and the manufactured goods need to be transported to market. So to produce _one unit of output each_, the industries require the following _inputs_ from each other :

$$
\begin{array}{ccc}
\text{MF} & \text{TR} & \text{AG}
\end{array}
$$

Consumption matrix

$$
C = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.5 & 0 & 0.2 \\ 0 & 0 & 0.2 \end{pmatrix}
\begin{matrix} \text{MF} \\ \text{TR} \\ \text{AG} \end{matrix}
$$

(require from — arrow pointing to MF column)

and so for a given output vector $\vec{x}$, we have the intermediate demand

$$
C\vec{x}.
$$

All told,

production vector

$$
\vec{x} = C\vec{x} + \vec{d}
$$

is the vector which tells how many units each industry needs to produce. As a linear system, this reads

$$
(\mathbb{I}_3 - C)\,\vec{x} = \vec{d}.
$$

To solve it, write either $\vec{x} = (I-C)^{-1}\vec{d}$, or row-reduce an augmented matrix. Row-reduction gives

$$
\left[\begin{array}{ccc|c} 0.6 & -0.3 & -0.3 & 60 \\ -0.5 & 1 & -0.2 & 10 \\ 0 & 0 & 0.8 & 40 \end{array}\right]
\rightarrow
\left[\begin{array}{ccc|c} 1 & -0.5 & -0.5 & 100 \\ -1 & 2 & -0.4 & 20 \\ 0 & 0 & 0.8 & 40 \end{array}\right]
\rightarrow
\left[\begin{array}{ccc|c} 1 & -0.5 & -0.5 & 100 \\ 0 & 1.5 & -0.9 & 120 \\ 0 & 0 & 0.8 & 40 \end{array}\right]
$$

$$
\Rightarrow x_3 = \frac{40}{0.8} = \boxed{50} \Rightarrow x_2 = \frac{2}{3}(0.9\cdot 50 + 120) = \frac{2}{3}(165) = \boxed{110} \Rightarrow x_1 = \frac{110+50}{2} + 100 = \boxed{180}
$$

— quite a bit bigger than $\vec{d}$. As for the inverse

matrix, we write

$$(\mathbb{I} - C)(\mathbb{I} + C + C^2 + C^3 + \cdots + C^m)$$

$$= \mathbb{I} + \underline{C + C^2 + \cdots + C^m}$$
$$\quad \underline{- C - C^2 + \cdots - C^m} - C^{m+1}$$

$$= \mathbb{I} - C^{m+1} \approx \mathbb{I} \quad (C^{m+1} \text{ eventually very small if sums along columns of } C \text{ are less than } 1)$$

$$\implies (\mathbb{I} - C)^{-1} \approx \mathbb{I} + C + C^2 + \cdots + C^m$$

$$\implies \vec{x} \approx \vec{d} + C\vec{d} + C^2\vec{d} + \cdots + C^m\vec{d} \quad (\text{for } m \text{ large}).$$

The book provides a conceptual explanation for this form of the solution.