Lecture 13 : More on Determinants

In lecture 12, we defined $\det : \left\{ \begin{matrix} n\times n \text{ real} \\ \text{matrices} \end{matrix} \right\} \longrightarrow \mathbb{R}$

$A \longmapsto \det(A) = \text{"}|A|\text{"}$

to be the unique function satisfying

(i) linearity in each row (with the other rows held fixed)

(ii) antisymmetry in the rows

(iii) $\det \mathbb{I}_n = 1$.

From these properties, we quickly deduced that

- if 2 rows are equal, then $\det A = 0$
- if A is upper/lower triangular, $\det A =$ product of diagonal entries (in particular, if A is diagonal, i.e. $A = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ then $\det A = d_1 \cdots d_n$)
- if $A_{\hat{i}\hat{j}} = (n-1)\times(n-1)$ matrix obtained by deleting i^{th} row + j^{th} column of A, $C_{ij} = (-1)^{i+j} \det A_{\hat{i}\hat{j}} = (i,j)^{\text{th}}$ cofactor, then (for any i)

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \Leftarrow \text{Laplace expansion along the } i^{\text{th}} \text{ row}.$$

Consequently,

- if A has a row of 0's, then $\det A = 0$.

Ex 1/

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 1\cdot \underbrace{\begin{vmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 3 & 0 \end{vmatrix}}_{A_{\hat{1}\hat{1}}} - 1\cdot \underbrace{\begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{vmatrix}}_{A_{\hat{1}\hat{4}}}$$

$$= 3\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 3(-3) - 1\cdot 1 + 1(9+1) + 2(-1-3) = -8. \quad //$$

Determinants and EROs (Elementary Row Operations)

Let $\mathcal{E}$ be an elementary matrix, so that $\tilde{A} = \mathcal{E} \cdot A$ is one row operation applied to A.

Theorem 1: If $\mathcal{E}$ is a $\left\{\begin{matrix} \text{replace} \\ \text{swap} \\ \text{scale by } \mu \end{matrix}\right\}$ operation, $\det \tilde{A} = \begin{cases} \det A \\ -\det A \\ \mu \det A. \end{cases}$

Proof: Write $A = \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{pmatrix}$.

$$\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j + \alpha \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = \underbrace{\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_n \end{pmatrix}}_{A} + \alpha \cdot \underbrace{\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}}_{0} = \det A$$

equal

by linearity

$$\det \begin{pmatrix} \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_i \\ \vdots \end{pmatrix} = -\det \begin{pmatrix} \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \end{pmatrix} = -\det A$$

by antisymmetry

$$\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \mu \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = \mu \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = \mu \det A.$$

by linearity

$\square$

Ex 2/ Find $\det\begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 14 & 10 \\ -2 & 6 & 10 & 33 \end{pmatrix}$.

Idea: row-reduce to an upper triangular using only replace & swap.

$$= \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 9 \end{vmatrix} = 1^3 \cdot 9 = 9.$$

Determinants and Invertibility

Theorem 2: $\det A \neq 0 \iff A$ invertible.

Proof: ($\Leftarrow$): A invertible $\Rightarrow A$ is obtained from I_n by EROs

$$\underset{\text{Thm 1}}{\Longrightarrow} \det A = \det I_n \cdot (-1)^{\#\text{swaps}} \Big/ \begin{pmatrix}\text{product of}\\ \text{scaling factors}\end{pmatrix} \neq 0.$$

($\Rightarrow$): $\det A \neq 0 \underset{\text{Thm 1}}{\Longrightarrow} \det(\operatorname{rref} A) = \det A \times (-1)^{\#\text{swap}} \times \begin{pmatrix}\text{product of}\\ \text{scaling factors}\end{pmatrix} \neq 0$

$\Rightarrow \operatorname{rref} A$ has no row of 0s

$\underset{A \text{ square}}{\Longrightarrow} \operatorname{rref} A = I_n$

$\Rightarrow A$ invertible. $\square$

Ex 3/ We know $\operatorname{rref}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \neq I_3$, so

$$\det\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} = 0.$$

//

Ex 4/ Find all values of α that make

$$A = \begin{pmatrix}\alpha&1&1\\\alpha&\alpha&1\\4&\alpha&\alpha\end{pmatrix}$$

<u>non</u>-invertible ("singular").

want $0 \Rightarrow$

$$\begin{vmatrix}\alpha&1&1\\\alpha&\alpha&1\\4&\alpha&\alpha\end{vmatrix} = \begin{vmatrix}\alpha&1&1\\0&\alpha-1&0\\4&\alpha&\alpha\end{vmatrix} \underset{\substack{\uparrow\\\text{expand in}\\\text{middle row}}}{=} (\alpha-1)\begin{vmatrix}\alpha&1\\4&\alpha\end{vmatrix} = (\alpha-1)(\alpha^2-4) = (\alpha-1)(\alpha-2)(\alpha+2)$$

$\Rightarrow$ values are $\alpha = 1, 2, -2$. //

Determinants and Products

Consider the elementary matrices once more. What are <u>their</u> determinants?

$$\begin{vmatrix}1&&&\\&\ddots&&\\&&\ddots&\\&\alpha&&\ddots\;1\end{vmatrix} = 1, \quad \begin{vmatrix}1&&&&\\&\ddots&&&\\&&0&\cdots&1\\&&1&\cdots&0\\&&&&\ddots\;1\end{vmatrix} = -1, \quad \begin{vmatrix}1&&&&\\&\ddots&&&\\&&1&&\\&&&\mu&\\&&&&1\\&&&&&\ddots\;1\end{vmatrix} = \mu$$

$\boxed{\text{Replace}}$ (upper or lower triangular with 1's on diagonal)

$\boxed{\text{Swap}}$ (get I_n by swapping i^{th} and j^{th} rows)

$\boxed{\text{Scale}}$

Looking at the statement of Theorem 1, we see it can be rephrased as follows:

(*) If E is an elementary matrix, then

$$\underbrace{\det(EA)}_{\text{result of applying ERO to } A} = \det(E)\cdot\det(A).$$

Theorem 3: Given $n\times n$ matrices A & B, $\det AB = (\det A)(\det B)$.

Proof: Case 1: $\det A = 0$. Then A isn't invertible.

If AB had an inverse C, then $\mathbb{I}_n = (AB)C = A(BC)$
$\Rightarrow BC$ is inverse to A, a contradiction. So
AB isn't invertible, and $\therefore \det AB = 0$.

Case 2: $\det A \neq 0$. Then A is invertible, and so may be written as a product of elementary matrices:

$$A = E_N \cdots E_1(\mathbb{I}_n)$$

$$\Rightarrow AB = E_N \cdots E_1 B.$$

By repeated application of (*),

$$\det A = \det E_N \cdots \det E_1 \cancel{\det \mathbb{I}_1}$$

& $\det AB = \det E_N \cdots \det E_1 \det B$, so $\det AB = \det A \cdot \det B$. $\square$

Corollary: If A is invertible, $\det(A^{-1}) = \dfrac{1}{\det A}$.

Ex 5/ $\det B^{-1}A^9B = \cancel{(\det B)^{-1}}(\det A)^9 \cancel{\det B} = (\det A)^9$. //

(What about $A+B$? In fact, we can't say anything about its determinant. It's certainly false that $\det A+B = \det A + \det B$; for example, take $A = \mathbb{I}_2$, $B = -\mathbb{I}_2$. Then $\det A + \det B = 1+1 = 2$, but $A+B$ is the zero matrix.)

Determinants and Column Vectors

I mentioned in Lecture 12 that if you define "det" to be the unique alternating multilinear normalised function on {columns of A} then you get the same function as doing it via rows (as we've done). Since transposing takes rows to columns, this statement is equivalent to

Theorem 4: $\det A = \det A^T$.

Proof: First, $\det A \neq 0 \iff \det A^T \neq 0$, since they are both invertible (with $(A^T)^{-1} = (A^{-1})^T$) or both not.

So if they're invertible, we have $A = E_1 \cdots E_N$ and $A^T = E_N^T \cdots E_1^T$. By Thm. 3, it suffices to check $\det E_i = \det E_i^T$. But this is obvious: swap & scale matrices are unchanged by transpose, and replace matrices have determinant 1. $\square$

Corollary: (i) Elementary column operations have the same effect on det as ERO's.

(ii) Laplace expansion holds for columns: for each j,

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} .$$

Ex 6/ If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, find $\begin{vmatrix} a-8g & 3b-24h & c-8i \\ d & 3e & f \\ g & 3h & i \end{vmatrix}$

$$\underset{\substack{\uparrow \\ \text{linearity in} \\ \text{2nd column}}}{=} 3 \begin{vmatrix} a-8g & b-8h & c-8i \\ d & e & f \\ g & h & i \end{vmatrix} \underset{\substack{\uparrow \\ \text{ERO} \\ \text{(replace)}}}{=} 3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \cdot 2 = 6 .$$

Ex 7/

$$\underbrace{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix}}_{\text{call this "}\Delta_n\text{"}} \underset{\substack{\uparrow \\ \text{replace}}}{=} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2 & \cdots & n-1 \end{vmatrix} \underset{\substack{\uparrow \\ \text{Laplace expansion} \\ \text{in 1st column}}}{=} 1 \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & n-1 \end{vmatrix}$$

$$= \Delta_{n-1} \underset{\text{similar argument}}{=} \cdots = \Delta_{n-2} = \cdots = \Delta_1 = 1$$

Determinants and Linear Independence

Recall from Lectures 11 and 8 that the following conditions on an $n \times n$ (square) matrix A are equivalent:

for the corresponding L.T.

(C1) A is invertible — T is invertible

(C2) The columns of A span $\mathbb{R}^n$ — T is onto

(C3) The columns of A are linearly independent — T is 1-1

The same goes for rows since columns of A are rows of A^T.

Theorem 5: $\det A \neq 0 \iff$ rows are lin. ind. & span $\mathbb{R}^n$

$\iff$ columns are lin. ind. & span $\mathbb{R}^n$.

Ex 8 / For what values of α is $\begin{pmatrix}\alpha\\ \alpha\\ 4\end{pmatrix}, \begin{pmatrix}1\\ \alpha\\ \alpha\end{pmatrix}, \begin{pmatrix}1\\ 1\\ \alpha\end{pmatrix}$ a linearly independent set?

We already solved this: They're independent $\iff$

$\det\begin{pmatrix}\alpha & 1 & 1\\ \alpha & \alpha & 1\\ 4 & \alpha & \alpha\end{pmatrix} \neq 0 \iff \alpha \neq 1, 2, -2.$

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