

Lecture 15: Vector Spaces

The position $f(t)$ of a metal ball on a spring with strength k , being driven by an oscillating magnetic field of amplitude F , and encountering friction measured by r , can be described by the (differential) equation

$$(1) \quad f''(t) + rf'(t) + kf(t) = F \cos(t).$$

Taking $r=0$ and $k=1$ for simplicity, we can rewrite (1) in terms of a linear differential operator T :

$$(2) \quad \underbrace{\left(\frac{d^2}{dt^2} + 1 \right)}_{=: T} f(t) = F \cos(t).$$

(why is it linear?)

I claim that — at least on the “space” of functions of the form

$$(3) \quad f(t) = x_1 \cos(t) + x_2 \sin(t) + x_3 t \cos(t) + x_4 t \sin(t)$$

— we can think of T as a linear transformation

$$T: \mathbb{R}^4 \rightarrow \mathbb{R}^4.$$

The idea is to identify a function (3) with

the vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, so that we get the table

vector $\vec{x} \in \mathbb{R}^4$	function $f(t)$	function $T f(t)$	vector $T \vec{x}$
\vec{e}_1	$\cos t$	0	$\vec{0}$
\vec{e}_2	$\sin t$	0	$\vec{0}$
\vec{e}_3	$t \cos t$	$-2 \sin t$	$-2 \vec{e}_2$
\vec{e}_4	$t \sin t$	$2 \cos t$	$2 \vec{e}_1$

$T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$

and T has standard matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The inhomogeneous ordinary differential equation, (2) becomes
(ODE)

$$A \vec{x}' = \begin{pmatrix} F \\ 0 \\ 0 \\ 0 \end{pmatrix} \longleftrightarrow \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} F \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with solution $x_4 = \frac{F}{2}$, $x_3 = 0$, x_1, x_2 free. So

the general solution to (2) is

$$f(t) = \underbrace{C_1 \cos(t) + C_2 \sin(t)}_{\text{solutions to homogeneous equation}} + \frac{F}{2} t \sin(t).$$

$$A\ddot{x} = \vec{0} / f'' + f = 0$$

The point of this example is that linear combinations (here, of functions) and linear transformations (here, $\frac{d^2}{dt^2} + 1$) occur throughout mathematics, and this makes it worth abstracting the properties of n -vectors in \mathbb{R}^n and linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^n$ to obtain more flexible concepts. (In particular, this will allow us to avoid constantly making identifications with \mathbb{R}^n .)



A (real) vector space, then, is a set V with a special element $\vec{0} \in V$, and three operations:

- ADDITION: $+ : V \times V \rightarrow V$
 $(\vec{v}, \vec{w}) \mapsto \vec{v} + \vec{w}$
- SCALAR MULTIPLICATION: $\cdot : \mathbb{R} \times V \rightarrow V$
 $(r, \vec{v}) \mapsto r\vec{v}$
- (ADDITIVE) INVERSION: $- : V \rightarrow V$
 $\vec{v} \mapsto -\vec{v}$

which satisfy the following properties:

(i) "+" is commutative and associative

$$(ii) \vec{0} + \vec{v} = \vec{v}$$

$$(i\ddot{a}) \vec{v} + (-\vec{v}) = \vec{0}$$

$$(iv) 1\vec{v} = \vec{v}$$

$$(v) r(s\vec{v}) = (rs)\vec{v}$$

$$(vi) \text{ distributivity: } \begin{cases} r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v} \\ (r+s)\vec{v} = r\vec{v} + s\vec{v} \end{cases}$$

The elements of V are called vectors.

Remarks on this definition:

$$(a) \quad 0\vec{v} = (0+0)\vec{v} \underset{(vi)}{=} 0\vec{v} + 0\vec{v} \underset{(iii)}{\implies} \vec{0} = 0\vec{v}$$

$$(b) \quad (-1)\vec{v} = -\vec{v} : \text{ because } \vec{v} + (-1)\vec{v} \underset{(iv)}{=} 1\vec{v} + (-1)\vec{v} \underset{(vi)}{=} (1+(-1))\vec{v} \\ = 0\vec{v} \underset{(a)}{=} \vec{0}$$

(now use (iii) & (i))

So we don't separately specify what is " $-\vec{v}$ " when constructing vector spaces: just the set V , and the operations "+" and "·".

Examples /

① $\mathbb{R}^m = m$ -tuples $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, "+" defined componentwise / etc.

② Directed line-segments in m -space (heretofore informally identified with ①), "+" defined by parallelogram law / etc.

③ $M_{m \times n} = m \times n$ matrices with real entries
" + " = matrix addition, " o " = scalar mult.
(matrix multiplication plays no role)

④ $\mathbb{P} =$ polynomials with real coefficients
" + " = addition of polynomials,
etc.

⑤ $\mathcal{F}(S, \mathbb{R}) =$ functions from a set S to \mathbb{R}
" + " adds values of functions pointwise: $(f+g)(s) = f(s) + g(s)$
" o " multiplies " " " " by scalar: $(rf)(s) = r f(s)$

e.g. $\mathcal{F}(\mathbb{N}, \mathbb{R}) =$ sequences

$\mathcal{F}(\mathbb{R}, \mathbb{R}) =$ functions of 1 variable



Non-Examples /

① \mathbb{R}^2 with "+" defined by $(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 - b_2)$
"·" " " " " $r(a_1, a_2) := (ra_1, ra_2)$.

What is the problem? $-(a, b)$ exists and $= (-a, b)$

But perhaps we should check distributivity:

$$(r + (-r))(a, b) = 0(a, b) = (0a, 0b) = (0, 0)$$

while

$$\begin{aligned} r(a, b) + (-r)(a, b) &= (ra, rb) + (-ra, -rb) = (ra + (-ra), rb - (-rb)) \\ &= (0, 2b) \neq (0, 0) ! \end{aligned}$$

In other words, (vi) fails if $b \neq 0, r \neq 0$.

So this isn't a vector space.

② \mathbb{R}^2 with $\begin{cases} (a_1, a_2) + (b_1, b_2) := (a_1 + b_1, 0) \\ r(a_1, a_2) := (ra_1, 0) \end{cases}$

Here the problem is (iv):

$$\text{if } b \neq 0, (a, b) \neq (a, 0) = 1(a, b).$$

Vector Subspaces

A subset $W \subseteq V$ of a vector space V is called a subspace if it is itself a (real) vector space, under the "+" and "." inherited from V . Rather than check all the properties again, we can invoke the

Proposition: A nonempty subset $W \subseteq V$ is a (vector) subspace if and only if it contains all linear combinations $a\vec{w}_1 + b\vec{w}_2$ of its own elements $\vec{w}_1, \vec{w}_2 \in W$ — that is, iff W is closed under addition & scalar multiplication.

The "proof" is that (i) - (vi) are essentially inherited from V (and W contains $\vec{0}$ b/c you can take the trivial linear combination $0\vec{w}_1 + 0\vec{w}_2$). [To me, the Proposition is a mini-Theorem; to the book, it's the definition.]

How to construct subspaces?

(A) Suppose we already have two: W_1 & W_2 , in V . Then

- $W_1 \cap W_2$ (the intersection) is one
- $W_1 \cup W_2$ (the union) is not — e.g. consider

$W_1 = x\text{-axis}$ and $W_2 = y\text{-axis}$ in \mathbb{R}^2 ;

the union doesn't contain $(1,0) + (0,1)$.



(B) To fix this, define

$$W_1 + W_2 := \{ \vec{w}_1 + \vec{w}_2 \mid \vec{w}_1 \in W_1, \vec{w}_2 \in W_2 \} \subseteq V.$$

(C) In the same spirit, given vectors $\vec{v}_1, \dots, \vec{v}_n \in V$, define

$$W := \text{span} \{ \vec{v}_1, \dots, \vec{v}_n \} = \{ a_1 \vec{v}_1 + \dots + a_n \vec{v}_n \mid a_1, \dots, a_n \in \mathbb{R} \} \subseteq V.$$

If $\text{span} \{ \vec{v}_i \}_{i=1}^n = V$ then we say "the \vec{v}_i span V ".

(D) Solutions, in \mathbb{R}^n , of $A \vec{x} = \vec{0}$ yield a subspace:

$$W = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n (= V \text{ here}).$$

[The same does not go for $A\vec{x} = \vec{b}$ (for some fixed $\vec{b} \neq \vec{0}$): If \vec{x}_1, \vec{x}_2 both solve $A\vec{x} = \vec{b}$, then does $\vec{x}_1 + \vec{x}_2$? No, it solves $A\vec{x} = 2\vec{b}$. So the solution set isn't closed under "+."]]

$$\textcircled{E} \quad C^0(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R}) \quad (\text{continuous functions})$$

$$P_n \subseteq P \quad (\text{polynomials of degree } \leq n)$$

$$\left\{ \begin{array}{l} \text{Upper triangular} \\ \text{matrices} \end{array} \right\} \subseteq M_{n \times n} .$$

The book has other interesting examples and you should have a look.