

Lecture 16: Kernel & Image

The last lecture illustrated how we can abstractify the properties of \mathbb{R}^m to obtain the notion of a vector space. This allows us to do away with the formality of identifying with \mathbb{R}^m when performing computations. The next step is to bring our notion of transformations up to this level.

Definition 1: A linear transformation is a map (function) between two (not necessarily different) vector spaces

$$T : V \rightarrow W$$

obeying linearity: $T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2)$.

(Note that $T(\vec{0}) = \vec{0}$ is an immediate consequence of this definition.)

Examples

① $V = \mathbb{R}^n, W = \mathbb{R}^m$

$T(\vec{v}) := A\vec{v}$ (we know this is linear already)

② $V = W = \mathbb{P}_3$ (polynomials of degree ≤ 3)

$T = \frac{d}{dt}$ is linear, since $\frac{d}{dt}(aP + bQ) = a \frac{dP}{dt} + b \frac{dQ}{dt}$

③ $V = W = C^0(\mathbb{R})$ (real-valued continuous functions on \mathbb{R})

T sends $f(x) \mapsto (Tf)(x) := \int_0^x f(t) dt$

④ $V = \mathbb{R}^4$, $W = \mathcal{F} :=$ space of functions spanned by $\cos(t), \sin(t), t\cos(t), t\sin(t)$

In Lecture 15, we had $T: \mathcal{F} \rightarrow \mathcal{F}$ given by $\frac{d^2}{dt^2} + 1$.

To make identifications with \mathbb{R}^4 , we need a different linear transformation, $S: \mathbb{R}^4 \rightarrow \mathcal{F}$ sending

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto x_1 \cos t + x_2 \sin t + x_3 t \cos t + x_4 t \sin t =: f_{\vec{x}}(t)$$

Two kinds of subspaces

Let $T: V \rightarrow W$ be a linear transformation.

Definition 2: The image (or range) of T is the set of all vectors in W that are "hit" by T :

$$\text{im}(T) := \{ \vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \} \subseteq W$$

Definition 3: The kernel of T is the set of vectors in V "killed" by T :

$$\text{ker}(T) := \{ \vec{v} \in V \mid T\vec{v} = \vec{0} \} \subseteq V.$$

Examples

① $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ has
 $\vec{x} \mapsto A\vec{x}$

$$\ker(T) = \{x \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} =: \underline{\text{Nul}}(A) \subseteq \mathbb{R}^n$$

null space of A

$$\text{im}(T) = \{\vec{b} \in \mathbb{R}^m \mid \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\} =: \underline{\text{Col}}(A) \subseteq \mathbb{R}^m$$

column space of A
= span of A's columns

② $\frac{d}{dt}: \mathbb{P}_3 \rightarrow \mathbb{P}_3$

$$\ker\left(\frac{d}{dt}\right) = \{P \in \mathbb{P}_3 \mid \frac{dP}{dt} = 0\} = \mathbb{P}_0 (= \mathbb{R}) \text{ is constant functions}$$

$$\text{im}\left(\frac{d}{dt}\right) = \{Q \in \mathbb{P}_3 \mid Q = \frac{dP}{dt} \text{ for some } P \in \mathbb{P}_3\} = \mathbb{P}_2$$

③ $\int_0^x = T: C^0(\mathbb{R}) \rightarrow C^0(\mathbb{R})$

$$\ker(T) = \{f \in C^0(\mathbb{R}) \mid \int_0^x f(t) dt \text{ is identically } 0 \text{ for all } x\} = \{0\}$$

$$\text{im}(T) = \{g \in C^0(\mathbb{R}) \mid g(x) = \int_0^x f(t) dt \text{ for some } f \in C^0(\mathbb{R})\} = \underline{C^1(\mathbb{R})}_0$$

(by the FTC)

continuously differentiable functions with $g(0)=0$

④ $S: \mathbb{R}^4 \rightarrow \sigma_{\vec{x}}$

$$\ker(S) = \{x \in \mathbb{R}^4 \mid f_{\vec{x}}(t) = 0 \text{ (for all } t)\} = \{\vec{0}\}$$

$$\text{im}(S) = \{f \in \sigma_{\vec{x}} \mid f = f_{\vec{x}} \text{ for some } \vec{x}\} = \sigma_{\vec{x}}$$



Let $T: V \rightarrow W$ be a linear transformation.

Proposition A: $\text{im}(T) \subseteq W$ and $\text{ker}(T) \subseteq V$
are subspaces.

Proof: If $w, w' \in \text{im}(T)$, then $\begin{cases} w = T(v) \\ w' = T(v') \end{cases}$ for some $v, v' \in V$.

By linearity, $T(av + bv') = aTv + bTv' = aw + bw'$
 $\Rightarrow aw + bw' \in \text{im}(T)$.

If $v, v' \in \text{ker}(T)$, then $\begin{cases} Tv = 0 \\ Tv' = 0 \end{cases} \Rightarrow$

$T(av + bv') = aTv + bTv' = a \cdot 0 + b \cdot 0 = 0 \Rightarrow$

$av + bv' \in \text{ker}(T)$. \square

Definition 4: (i) T is onto if $\text{im}(T) = W$

(ii) T is 1-1 if $v \neq v' \Rightarrow Tv \neq Tv'$
(i.e. $Tv = Tv' \Rightarrow v = v'$)

Proposition B: T is 1-1 $\Leftrightarrow \text{ker}(T) = \{0\}$.

Proof: (\Rightarrow) is clear: only 0 can go to 0.

(\Leftarrow) Suppose $\text{ker}(T) = \{0\}$, and let $Tv = Tv'$.

Then by linearity, $0 = Tv - Tv' = T(v - v')$ so

$v - v' \in \text{ker}(T) (= \{0\}) \Rightarrow v - v' = 0 \Rightarrow v = v'$. \square

Examples /

- ① T is 1-1 $\Leftrightarrow \text{Nul}(A) = \{0\}$; T is onto $\Leftrightarrow \text{Col}(A) = \mathbb{R}^m$
 - ② $\frac{d}{dt}$ is neither 1-1 nor onto
 - ③ T is 1-1 but not onto
 - ④ S is 1-1 and onto
- //

Two kinds of problems

How do we find Null and Column spaces of a matrix A ?

- ① Let $A = \begin{pmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$. (a) Is $\vec{v} = \begin{pmatrix} 6 \\ 2 \\ 5 \\ 1 \\ 0 \end{pmatrix}$ in $\text{Nul}(A)$?
(b) Parametrically describe $\text{Nul}(A)$.

(a) Yes: just compute $A\vec{v} = \vec{0}$

(b) We want all the solutions of $A\vec{x} = \vec{0}$, and we know how to describe them parametrically: x_2, x_4 are non-pivot variables and thus free; we have

$$\begin{cases} x_1 = 4x_2 - 2x_4 \\ x_3 = 5x_4 \\ x_5 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4x_2 - 2x_4 \\ x_2 \\ 5x_4 \\ x_4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{Nul}(A) = \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} \mid x_2, x_4 \in \mathbb{R} \right\} \subseteq \mathbb{R}^5. \\ \text{(i.e. the span)}$$

So it's easy to check if a given vector is in $\text{Nul}(A)$,
but requires work to construct vectors in $\text{Nul}(A)$.

(II) Let $A = \begin{pmatrix} 6 & -4 \\ -3 & 1 \\ -9 & 6 \\ 9 & -6 \end{pmatrix}$. (a) Is $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ in $\text{Col}(A)$?
(b) Parametrically describe $\text{Col}(A)$.

(a) Need to see if $A\vec{x} = \vec{v}$ is consistent.

$$\left[\begin{array}{cc|c} 6 & -4 & 0 \\ -3 & 1 & 1 \\ -9 & 6 & 0 \\ 9 & -6 & 0 \end{array} \right] \xrightarrow{\text{3 row ops}} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ -3 & 1 & 1 \\ -3 & 2 & 0 \\ -3 & 2 & 0 \end{array} \right] \xrightarrow{\text{3 replace ops.}} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

which yields $\begin{cases} -x_2 = 1 \\ 3x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1 \end{pmatrix}$.

So, Yes.

(b) easy: $\text{Col}(A) = \left\{ s \begin{pmatrix} 6 \\ -3 \\ -9 \\ 9 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \\ 6 \\ -6 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$

In contrast to $\text{Nul}(A)$, it's easy to construct vectors in $\text{Col}(A)$,
but requires work to check if a given vector is in $\text{Col}(A)$.