Lecture 17: Basis of a vector space

Let V be a vector space. A finite sequence $\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$ is linearly dependent if $\exists\, c_1, \dots, c_k \in \mathbb{R}$, not all 0, such that

(*) $$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}.$$

Otherwise, it is linearly independent. Linear independence is equivalent to the following property (why??): no $\vec{v}_j$ is a linear combination of the $\vec{v}_1, \dots, \vec{v}_{j-1}$ preceding it.

Ex 1/ In $\mathbb{P}_3$, $\{t, t^2\}$ is independent, but $\{t, t(t-2), t^2\}$ is dependent because $(-2)t + (-1)t(t-2) + (1)t^2 = 0$. //

Notice that if a linear dependence relation (*) holds in V, and $T: V \to W$ is a linear transformation, then

$$c_1T(\vec{v}_1) + \dots + c_kT(\vec{v}_k) = \vec{0}$$

holds in W. So we arrive at the

Observation: If $T(\vec{v}_1), \dots, T(\vec{v}_k)$ are independent, then so are $\vec{v}_1, \dots, \vec{v}_k$.

Ex 2/ In $C^0(\mathbb{R})$, $\{\cos(t), \sin(t), t\cos(t), t\sin(t)\}$ is independent.

OK, how would you prove that? Maybe take values of the functions at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$: that gives a linear transformation

$$F : C^0(\mathbb{R}) \to \mathbb{R}^4$$

sending

$$\cos(t) \longmapsto \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\sin(t) \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$t\cos(t) \longmapsto \begin{pmatrix} 0 \\ 0 \\ -\pi \\ 0 \end{pmatrix}$$

$$t\sin(t) \longmapsto \begin{pmatrix} 0 \\ \pi/2 \\ 0 \\ -3\pi/2 \end{pmatrix}.$$

You can check that the 4 vectors on the right are independent in $\mathbb{R}^4$. Therefore, the functions are independent in $C^0(\mathbb{R})$, by the Observation. //

We have made a lot of use lately of the vectors $\vec{e}_1, \dots, \vec{e}_n$ in $\mathbb{R}^n$. Any vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$ <u>can be written uniquely</u> as a linear combination of them: $x_1\vec{e}_1 + \dots + x_n\vec{e}_n$. But $\vec{e}_1, \dots, \vec{e}_n$, while especially <u>convenient</u>, are far from being the only set of n vectors with this property. In fact, the vectors on the right in Example 2 also have this property.

<u>Definition</u>: A finite sequence $\{\vec{v}_1, \dots, \vec{v}_k\} \subset V$ is called a <u>basis of V</u> if (a) it is linearly independent and (b) it spans V.

Given a basis of V, any larger collection $\{\vec{v}_1, \ldots, \vec{v}_k; \vec{v}'\}$ is dependent: since $\vec{v}_1, \ldots, \vec{v}_k$ span V, $\vec{v}' = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k$

$\Rightarrow 0 = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k + (-1)\vec{v}'$ is a dependence relation.

Ex 3/ (1) $V = \mathbb{R}^n$ has basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$, called the standard basis.

(2) $V = \mathbb{P}_3$ has basis $\{1, t, t^2, t^3\}$.

(3) $V = \mathbb{P}_2(x,y)$ (= polynomials in x, y of degree ≤ 2) has basis $\{1, x, y, xy, x^2, y^2\}$. //

Ex 4/ If A is an invertible $n \times n$ matrix, its columns $\{\vec{v}_1, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^n$. [True for rows too, since A^T is invertible if A is.] Why? Well, we need to check (a) and (b) in the definition:

(a): $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0} \Longrightarrow A\vec{x} = \vec{0} \underset{\text{apply } A^{-1}}{\Longrightarrow} \vec{x} = \vec{0}$.

So $\vec{v}_1, \ldots, \vec{v}_n$ is linearly independent.

(b): let $\vec{u} \in \mathbb{R}^n$, set $\vec{x} := A^{-1}\vec{u}$. Then $A\vec{x} = \vec{u}$, so $\vec{u}$ is in the span of A's columns. Since $\vec{u}$ was arbitrary, A's columns span $\mathbb{R}^n$. //

Our notion of bases in this course is finite — we won't deal with infinite bases. So when does a vector space have a (finite) basis?

Ex 5/ $C^0(\mathbb{R})$ and $\mathbb{P}$ do not. //

Ex 6/ Call V finitely generated if some finite subset $\mathcal{S} = \{\vec{v}_1, \dots, \vec{v}_n\}$ spans V. In this case, some subset of $\mathcal{S}$ is a basis for V.* Why? Well, you can go through $\mathcal{S}$ throwing out any $\vec{v}_i$ that is a linear combination of the vectors preceding it, until this is no longer the case. To see that the resulting subset still spans V, note that if (say) $\vec{v}_k = \alpha_1 \vec{v}_1 + \dots + \alpha_{k-1}\vec{v}_{k-1}$, then any linear combination $a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ can be rewritten as a linear combination

$$(a_1 + a_k\alpha_1)\vec{v}_1 + \dots + (a_{k-1} + a_k\alpha_{k-1})\vec{v}_{k-1} + a_{k+1}\vec{v}_{k+1} + \dots + a_n\vec{v}_n$$

in which $\vec{v}_k$ is omitted. So removing $\vec{v}_k$ won't affect the span. The new subset is linearly independent by the criterion on p. 1. //

* The book calls this the "Spanning Set Theorem".

Now let A be an $m \times n$ matrix.

Consider $\mathrm{Nul}(A) \subset \mathbb{R}^n$, the null space of A. Can we construct a basis?

Ex 7/ $A = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \rightsquigarrow \mathrm{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

x_2, x_4 free

$$\begin{aligned}
\mathrm{Nul}(A) &:= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad (\subset \mathbb{R}^5) \\
&= \left\{ \begin{pmatrix} -2x_2 - 3x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{pmatrix} \,\middle|\, x_2, x_4 \in \mathbb{R} \right\} \\
&= \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, x_2, x_4 \in \mathbb{R} \right\} \\
&= \mathrm{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\},
\end{aligned}$$

and in fact these 2 5-vectors form a basis. //

The nature of the procedure in this example guarantees a basis, b/c : (a) the vectors span $\mathrm{Nul}(A)$; and (b) there is one vector for each free variable, and if that variable is x_k, then

$\left\{\begin{array}{l}\text{that vector has } k^{\text{th}} \text{ entry } 1 \\ \text{the other vectors have } k^{\text{th}} \text{ entry } 0\end{array}\right. \Rightarrow$ the k^{th} vector is <u>NOT</u> a linear combination of the others (for each k)

$\Rightarrow$ these vectors are independent.

<u>Remark</u>: Part of what makes this work is that row-reduction doesn't affect the null space:

$$\mathrm{Nul}\,(\mathrm{rref}(A)) = \mathrm{Nul}(A).$$

The same is <u>not</u> true for the column space:

$$\mathrm{Col}\,(\mathrm{rref}(A)) \neq \mathrm{Col}(A).$$

So how do we get a basis of $\mathrm{Col}(A)$?

Suppose $\mathcal{E}$ is a product of elementary matrices which reduce A to RREF:

$$\begin{pmatrix} \uparrow & & \uparrow \\ \vec{w}_1 & \cdots & \vec{w}_n \\ \downarrow & & \downarrow \end{pmatrix} = \mathrm{rref}\,(A) = \mathcal{E}\cdot A = \mathcal{E}\begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \mathcal{E}\vec{v}_1 & \cdots & \mathcal{E}\vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}.$$

Since $\mathcal{E}$ is invertible, $\vec{w}_1, \dots, \vec{w}_n$ have exactly the same dependence relations as $\vec{v}_1, \dots, \vec{v}_n$. That means that if all the $\vec{w}$'s can be written as linear combinations of some subset $\vec{w}_{i_1}, \dots, \vec{w}_{i_\ell}$ — i.e. this smaller set spans $\mathrm{Col}(\mathrm{rref}(A))$ — then all the $\vec{v}$'s can be written as linear combinations of $\vec{v}_{i_1}, \dots, \vec{v}_{i_\ell}$ — these span $\mathrm{Col}(A)$. Since the columns of rref(A) with leading 1's evidently span Col(rref(A)) (why?), that means the pivot columns in A span Col(A). Similarly, the pivot columns in rref(A) are clearly independent (they are a subset of $\vec{e}_1, \dots, \vec{e}_m$!), so therefore the pivot columns in A are independent. This proves the

THEOREM: The pivot columns of A form a basis for the column space Col(A).

Ex 8/ A as in Example 7, $\text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

pivot columns

$\Rightarrow$ a basis for Col(A) is

$$\left\{ \begin{pmatrix} 0 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \\ 6 \end{pmatrix} \right\},$$

the 1st, 3rd, & 5th columns of A. //

Upshot: We can find bases of Nul(A) and Col(A) easily by row-reducing A to RREF.