

Lecture 18: Coordinates w.r.t. a basis

Coordinate Vectors

Let V be a vector space, and $B = \{\vec{b}_1, \dots, \vec{b}_n\} \subset V$ a basis. Take any $\vec{x} \in V$.

Theorem 1: We can write \vec{x} as a sum $\sum \alpha_i \vec{b}_i$ in exactly one way.

Proof: Since B spans V , we can write \vec{x} as such a sum.

If $\sum_{i=1}^n \alpha_i \vec{b}_i = \vec{x} = \sum_{i=1}^n \gamma_i \vec{b}_i$, then $\vec{0} = \sum_{i=1}^n (\alpha_i - \gamma_i) \vec{b}_i$. Since

B is a linearly independent set, $\alpha_i - \gamma_i = 0$ for all i ; that is, $\alpha_i = \gamma_i$. □

We write $[\vec{x}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} =$ the coordinate vector of \vec{x} with respect to B .

Notice that the procedure of "taking the coordinate vector" gives a map

$$[\cdot]_B : V \rightarrow \mathbb{R}^n \\ \vec{x} \mapsto [\vec{x}]_B.$$

Theorem 2: $[\cdot]_B$ is a linear transformation which is also 1-1 and onto, i.e. an isomorphism.

Proof: If $\vec{x} = \sum_{i=1}^n \alpha_i \vec{b}_i$ & $\vec{y} = \sum_{i=1}^n \beta_i \vec{b}_i$, then $c\vec{x} = \sum c\alpha_i \vec{b}_i$
 & $\vec{x} + \vec{y} = \sum (\alpha_i + \beta_i) \vec{b}_i$.

$$\left. \begin{aligned} \text{So } [c\vec{x}]_{\mathcal{B}} &= \begin{pmatrix} c\alpha_1 \\ \vdots \\ c\alpha_n \end{pmatrix} = c \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = c [\vec{x}]_{\mathcal{B}} \\ \text{and } [\vec{x} + \vec{y}]_{\mathcal{B}} &= \begin{pmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} \end{aligned} \right\} \Rightarrow (\cdot)_{\mathcal{B}} \text{ linear.}$$

If $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, then $\vec{x} = 0\vec{b}_1 + \dots + 0\vec{b}_n = \vec{0} \Rightarrow (\cdot)_{\mathcal{B}}$ is 1-1

Any $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is of the form $[\alpha_1 \vec{b}_1 + \dots + \alpha_n \vec{b}_n]_{\mathcal{B}} \Rightarrow (\cdot)_{\mathcal{B}}$ is onto. \square

Linear transformations which are isomorphisms $T: V \xrightarrow{\cong} W$ is the notation have inverses which are also linear transformations. They therefore produce an exact correspondence between linear combinations, linear independence/dependence, etc. in the two vector spaces.

Ex 1 / $[\cdot]_{\mathcal{B}}: \mathbb{P}_h \xrightarrow{\cong} \mathbb{R}^{h+1}$, $\mathcal{B} = \{1, t, \dots, t^h\}$.

$$a_0 + a_1 t + \dots + a_h t^h \mapsto \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_h \end{pmatrix}$$

Ex 2 / $[\cdot]_{\mathcal{B}}: W \xrightarrow{\cong} \mathbb{R}^4$, where $W = \text{span}\{c \cos t, s \sin t, c \cos t, -s \sin t\}$

$$\begin{aligned} a \cos t + b \sin t \\ + c \cos t + d \sin t \end{aligned} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

\mathcal{B}

Ex 3 / $A = m \times n$ matrix with k non-pivot columns.

$$V = \text{Nul}(A) \subset \mathbb{R}^n$$

$B = \{\vec{v}_1, \dots, \vec{v}_k\}$ the basis of V produced by our procedure in Lecture 17

$$[\cdot]_B: V \xrightarrow{\cong} \mathbb{R}^k$$

More concretely, the plane in $\mathbb{R}^{3(=n)}$ described by $4x_1 - 5x_2 + 2x_3 = 0$ is a subspace of this type, with

$$A = \begin{bmatrix} 4 & -5 & 2 \end{bmatrix}.$$

We have $B = \left\{ \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5/4 \\ 1 \\ 0 \end{pmatrix} \right\}$, and $[\cdot]_B$ gives

an isomorphism to \mathbb{R}^2 . //

Bases of \mathbb{R}^n

Any basis $B = \{\vec{b}_1, \dots, \vec{b}_k\} \subset \mathbb{R}^n$

has an associated $n \times k$ matrix

$$P_B := \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \dots & \vec{b}_k \\ \downarrow & & \downarrow \end{pmatrix}.$$

Since B is linearly independent, $\text{rref}(P_B)$ has a leading 1 in each column $\implies k \leq n$.

Since B spans \mathbb{R}^n , $\text{rank}(P_B)$ has a leading 1 in each row $\implies k \geq n$.

So $k = n$, and since the columns of P_B span \mathbb{R}^n , P_B is invertible.

What does this matrix have to do with anything, though?

Ex 4 / $B = \left\{ \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \end{pmatrix} \right\}$, $[\vec{x}]_B = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$. What is \vec{x} ?

$$\vec{x} = 6\vec{b}_1 + 4\vec{b}_2 = 6 \begin{pmatrix} 5 \\ -5 \end{pmatrix} + 4 \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 50 \\ -36 \end{pmatrix}, \text{ done.}$$

$$\text{But notice: } \vec{x} = \begin{pmatrix} 5 & 5 \\ -5 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = P_B [\vec{x}]_B. //$$

Indeed, if $[\vec{x}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, then by definition $\vec{x} = \alpha_1 \vec{b}_1 + \dots + \alpha_n \vec{b}_n = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \dots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = P_B [\vec{x}]_B.$

In fact, " \vec{x} " is itself a coordinate vector — with respect to the standard basis $e = \{\vec{e}_1, \dots, \vec{e}_n\}$:

$$\vec{x} = [\vec{x}]_e.$$

So we call P_B the change-of-basis (or change-of-coordinates) matrix from B to e .

