Lecture 19 : Dimension

Let's agree to call a vector space V finitely generated or finite-dimensional if it is spanned by a finite set of vectors. Otherwise we'll call it infinite dimensional.

In this case it can't have a basis in our sense (which is finite).

Theorem 1: A finite-dimensional vector space has a basis, and any two bases for V have the same number of elements.

Proof: By the "spanning set theorem" (Lecture 17, Example 6), V has a basis $\{\vec{b}_1, \ldots, \vec{b}_n\} = \mathcal{B}$. Let $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_m\} \subset V$ be any other basis, and consider its image $\{[\vec{u}_1]_{\mathcal{B}}, \ldots, [\vec{u}_m]_{\mathcal{B}}\}$ under the isomorphism $[\cdot]_{\mathcal{B}} : V \xrightarrow{\cong} \mathbb{R}^n$. I claim that this is a basis of $\mathbb{R}^n$. Since any basis of $\mathbb{R}^n$ has n elements (Lecture 18, pp. 3-4), $m = n$ (and we are done).

To check the claim:

- Let $\vec{x} \in \mathbb{R}^n$. As $[\cdot]_{\mathcal{B}}$ is onto, $\vec{x} = [\vec{w}]_{\mathcal{B}}$ for some $\vec{w} \in V$. Since $\mathcal{U}$ is a basis, we can write $\vec{w} = \sum_{i=1}^m \gamma_i \vec{u}_i$. So $\vec{x} = [\vec{w}]_{\mathcal{B}} = \sum_{i=1}^m \gamma_i [\vec{u}_i]_{\mathcal{B}}$; conclude that the $[\vec{u}_i]_{\mathcal{B}}$ span $\mathbb{R}^n$.
- Suppose $0 = \sum_{i=1}^m c_i [\vec{u}_i]_{\mathcal{B}}$. Then $0 = \left[\sum_{i=1}^m c_i \vec{u}_i\right]_{\mathcal{B}}$, and since $[\cdot]_{\mathcal{B}}$ is 1-to-1, $0 = \sum_{i=1}^m c_i \vec{u}_i$. Since $\mathcal{U}$ is a basis, the $c_i = 0$. Conclude that the $[u_i]_{\mathcal{B}}$ are independent. $\square$

We can now make the key

<u>Definition</u>: The number of elements in a basis for V is called its dimension, written $\dim(V)$.

By the same approach as in the proof of Theorem 1 (using $[\cdot]_{\mathcal{B}}$ to identify with $\mathbb{R}^n$), we see that:

- less than $n = \dim(V)$ vectors can't span V; and
- more than $n = \dim(V)$ vectors can't be independent in V.

Ex 1/ $\dim \mathbb{R}^n = n$ //

Ex 2/ $\dim \mathbb{P}_k = k+1$ //

count the # of elements in the standard basis

Ex 3/ In $\mathbb{R}^3$, we have the subspaces $\{\vec{0}\}$ (0-dim'l), $\operatorname{span}\{\vec{e}_1\}$ (1-dim'l), $\operatorname{span}\{\vec{e}_1, \vec{e}_2\}$ (2-dim'l). More generally, the 1-dim'l subspaces are lines thru $\vec{0}$, & the 2-dim'l ones are the planes thru $\vec{0}$. //

Ex 4/ What is dim of $W = \left\{ \begin{pmatrix} p-2q \\ 2p-9q+5r \\ -2q+2r \\ -3p+6r \end{pmatrix} \middle| \; p,q,r \in \mathbb{R} \right\} \overset{\text{subspace}}{\subset} \mathbb{R}^4$?

$$W = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix} \right\} = \operatorname{Col}\underbrace{\begin{pmatrix} 1 & -2 & 0 \\ 2 & -9 & 5 \\ 0 & -2 & 2 \\ -3 & 0 & 6 \end{pmatrix}}_{A}.$$

To find a basis, need to know the pivot columns:

$$\operatorname{rref}(A) = \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{basis of } W = \operatorname{Col}(A) \text{ is}$$

pivot columns

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -9 \\ -2 \\ 0 \end{pmatrix} \right\}.$$

So $\dim(W) = 2$. //

Theorem 2: Let $W \subseteq V$ be a subspace (of a finite dimensional vector space V). Then $\dim W \leq \dim V$, with equality if and only if $W = V$.

Proof: Let $\mathcal{B}_0 = \{\vec{b}_1, \ldots, \vec{b}_k\} \subset W$ be a basis. Unless $W = V$, it doesn't span V. Let $\vec{b}_{k+1} \in V$ be outside $W = \mathrm{span}(\mathcal{B}_0)$; then $\{\vec{b}_1, \ldots, \vec{b}_k; \vec{b}_{k+1}\}$ is independent. If it still doesn't span V, we can add $\vec{b}_{k+2}$ and so on. The process has to terminate b/c otherwise V would accomodate an infinite independent set, contradicting its finite-dimensionality. When it terminates (at some n), the set spans V. So $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k, \vec{b}_{k+1}, \ldots, \vec{b}_n\}$ is a basis for V containing $\mathcal{B}_0$, and $\dim W = k \leq n = \dim V$.

If $k = n$ then we are in the case where $\mathcal{B}_0$ already spanned V, i.e. $W = V$. $\square$

Remarks: ① The proof shows that any basis of W can be extended to one of V.

② Suppose we know $\dim(V) = n$, and

$$S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset V.$$

Then

(a) S independent $\Rightarrow$ S is a basis of V.
(automatically spans V)

(b) S spans V $\Rightarrow$ S is a basis of V
(automatically lin. ind.).

So it suffices to check independence OR spanning V.

Ex 5/ Is $\{1, 2t, -2+4t^2, -12t+8t^3\}$ a basis of $\mathbb{P}_3$?

Only need to check that they <u>span</u> $\mathbb{P}_3$, for which it suffices to show that you can get $1, t, t^2, t^3$ as linear combinations of them. (This is left to you.) //

Ex 6/ What are the dimensions of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$ for

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} ?$$

Row-reduce $\rightsquigarrow$ $\mathrm{rref}(A) = \begin{pmatrix} \boxed{1} & 2 & 0 & 3 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The pivot columns of A, i.e. the 1^{st}, 3^{rd}, & 5^{th} columns (of A, not $\mathrm{rref}(A)$) give a basis of $\mathrm{Col}(A)$.
So <u>$\dim(\mathrm{Col}(A)) = 3$</u>.

The non-pivot columns of A correspond to the free variables x_2 & x_4, associated to each of which you have a vector in the basis $\left\{ \begin{pmatrix} -2 \\ \boxed{1} \\ 0 \\ \boxed{0} \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ \boxed{0} \\ 1 \\ \boxed{1} \\ 0 \end{pmatrix} \right\}$ of $\mathrm{Nul}(A)$. So <u>$\dim(\mathrm{Nul}(A)) = 2$</u>. //

More or less the same words prove the following general

<u>**Theorem 3**</u>: $\dim(\mathrm{Nul}(A)) = \#$ of non-pivot columns

$\dim(\mathrm{Col}(A)) = \#$ of pivot columns.