

Lecture 2: Row reduction

Using the "Replace", "Swap", and "Scale" row-operations, we shall now discuss how to put any matrix in a particularly nice form:

Reduced Row-Echelon Form (RREF)

A matrix A is in RREF if all of the following hold:

- (i) the first nonzero entry of each row is 1, called a "leading 1";
- (ii) when a column contains a leading 1, all other entries in that column are 0 (this is called a "pivot column"); and
- (iii) when a row contains a leading 1, each row above it contains a leading 1 to the left.

The weaker notion of Row-Echelon Form (REF) is obtained by dropping (i) and weakening (ii) and (iii):

- (ii') when a column contains the first nonzero entry of some row, all the entries of the column

below it are 0 ; and

(iii') the leading nonzero entry of a row occurs further to the right than all the leading entries in the rows above it.

Ex 1 / If "*" stands for "arbitrary numbers",
then

$$\begin{pmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ are in}$$

reduced row-echelon form (RREF),

while (if "." stands for "arbitrary NONZERO number")

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \end{pmatrix} \text{ are in}$$

row-echelon form (REF).

Row reduction means "some procedure for associating an RREF matrix to a given matrix A ":

$$A \mapsto \text{rref}(A).$$

Since the procedure simply applies row operations to A , the new matrix is row-equivalent to A .

FACT: There is exactly one RREF matrix row-equivalent to a given matrix A .

(We'll explain why in Lecture 3.)

CONSEQUENCE: $\text{rref}(A)$ is independent of the procedure used!

The book has a 2-stage row-reduction algorithm:

- convert A to a REF matrix
- convert the REF matrix to a RREF one.

(See the "appendix" below.) Here's a simpler version:

Row-reduction algorithm

(a) "cursor" starts at upper left-hand entry of matrix;

(b) move cursor to right if the cursor entry and all entries below it are 0; repeat until this is no longer the case;

(c) if cursor entry = 0, swap cursor row with the first row below it having nonzero entry in the cursor column;

SWAP

(d) divide cursor row by cursor entry;

SCALE

(e) eliminate all other entries in the cursor column by adding suitable multiples of the cursor row to other rows;

REPLACE

(f) if cursor is at bottom right, STOP.

Otherwise, move down & to the right, and go back to (b).

Let's illustrate with an example: $\boxed{} = \text{cursor}$

Ex 2/ $A = \begin{pmatrix} \boxed{0} & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(c)} \begin{pmatrix} \boxed{2} & 4 & 2 & 4 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 3 & 3 & 3 \\ 3 & 6 & 6 & 3 & 6 \end{pmatrix}$

$\xrightarrow{(e)} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & \boxed{0} & 1 & -1 & -1 \\ 0 & \boxed{0} & 1 & -1 & -1 \\ 0 & 0 & 3 & -3 & 3 \end{pmatrix} \xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & \boxed{0} & \boxed{2} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix}$

$\xrightarrow{(e)} \begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix} = \text{rref}(A).$

Now, how do we use this to solve a linear system?

STEP 1: Convert the system to an augmented matrix

$$M = [A \mid \vec{b}]$$

STEP 2: Apply the above algorithm to compute $\text{rref}(M)$.

STEP 3: Convert back to a linear system and find the tuples $(x_1, \dots, x_n)$ solving it. (More precisely: use the non-pivot variables to parameterize the solution set. This is sometimes called "back substitution".)

Ex 3/

$$\begin{cases} 3x_1 - 6x_2 + 2x_3 - x_4 = 1 \\ -2x_1 + 4x_2 + x_3 + 3x_4 = 4 \\ x_3 + x_4 = 2 \\ x_1 - 2x_2 + x_3 = 1 \end{cases}$$

STEP 1

$$M = \left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 1 \end{array} \right] \xrightarrow[\substack{\text{scale} \\ P_1 \leftrightarrow \frac{1}{3}P_1}]{\substack{\text{scale} \\ P_1 \leftrightarrow \frac{1}{3}P_1}} \left[\begin{array}{cccc|c} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{7}{3} & \frac{7}{3} & \frac{11}{3} \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array} \right] \xrightarrow[\substack{\text{scale} \\ P_2 \leftrightarrow \frac{3}{7}P_2}]{\substack{\text{replace} \\ \begin{cases} P_2 \leftrightarrow P_2 + 2P_1 \\ P_3 \leftrightarrow P_3 - P_1 \\ P_4 \leftrightarrow P_4 - P_1 \end{cases}}} \left[\begin{array}{cccc|c} 1 & -2 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array} \right] \quad \text{STEP 2}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow[\substack{\text{replace} \\ \begin{cases} P_3 \leftrightarrow P_3 - P_2 \\ P_4 \leftrightarrow P_4 - \frac{1}{3}P_2 \\ P_1 \leftrightarrow P_1 - \frac{2}{3}P_2 \end{cases}}]{\substack{\text{replace} \\ \begin{cases} P_3 \leftrightarrow P_3 - P_2 \\ P_4 \leftrightarrow P_4 - \frac{1}{3}P_2 \\ P_1 \leftrightarrow P_1 - \frac{2}{3}P_2 \end{cases}}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] = \text{ref}(M)$$

STEP 3

$$\begin{cases} x_1 - 2x_2 - x_4 = -1 \\ x_3 + x_4 = 2 \end{cases}$$

Solve for the variables corresponding to pivot columns, called basic variables:

$$\begin{cases} x_1 = 2x_2 + x_4 - 1 \\ x_3 = 2 - x_4 \end{cases}$$

The free variables are the non-pivot ones, and are so named because they can be freely chosen. They parametrize the solution set of the linear system:

$$\mathcal{S} = \{(2x_2 + x_4 - 1, x_2, 2 - x_4, x_4) \mid x_2, x_4 \in \mathbb{R}\} //$$

Ex 4 /

$$\begin{cases} 3x_1 - 6x_2 + 2x_3 - x_4 = 1 \\ -2x_1 + 4x_2 + x_3 + 3x_4 = 4 \\ x_3 + x_4 = 2 \\ x_1 - 2x_2 + x_3 = 0 \end{cases}$$

← same as Ex. 3 except for this

$$\left[\begin{array}{cccc|c} 3 & -6 & 2 & -1 & 1 \\ -2 & 4 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -2 & 1 & 0 & 0 \end{array} \right] \xrightarrow[\text{Ex. 3}]{\text{steps from}} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow[\substack{\text{scale} \\ p_3 \leftrightarrow -1 \cdot p_3}]{\text{swap } p_3 \leftrightarrow p_4} \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow[\substack{\text{replace} \\ p_1 \leftrightarrow p_1 + p_3 \\ p_2 \leftrightarrow p_2 - 2p_3}]{\text{scale}} \left\{ \begin{array}{l} x_1 - 2x_2 - x_4 = 0 \\ x_3 + x_4 = 0 \\ \boxed{0 = 1} \end{array} \right.$$

Solution set

$$\mathcal{S} = \emptyset$$

System is inconsistent! //

From the last example, we notice that

- If the last column of the augmented matrix is a pivot column, then the system is inconsistent.

Otherwise, steps 1-3 tell us how to solve the system, and so:

- If the last column is non-pivot, the system is consistent; it has a unique solution if there are no free variables, i.e. if all but the last column are pivots.

So far we have looked at m equations in n unknowns, with $m = n$. What about the other cases?

- If $m > n$, the system is called overdetermined.
- If $m < n$, the system is called underdetermined.

Q: Can an overdetermined system be consistent?

A: YES. But some equations will have to be "linear combinations" of others.

Q: Can an underdetermined system have a unique solution?

A: NO. Think about intersections of planes in space: two planes in 3-space will never intersect in a point.

A more mathematical answer can be formulated as follows:

When you reduce an augmented matrix of the form

A diagram showing an augmented matrix represented as a square with a vertical line on the right side. A blue double-headed vertical arrow to the left of the square is labeled 'm', indicating the number of rows. A blue double-headed horizontal arrow above the square is labeled 'n', indicating the number of columns. To the right of the square, the text $(m < n)$ is written in blue.

to RREF, there must be non-pivot columns, hence free variables. This is because each pivot column contains a leading 1 for some row, and there are only m rows hence $\leq m$ leading 1's. In general, the number of pivots is at most the smaller of m & n .

APPENDIX (The book's row-reduction algorithm)

PART I Produce REF matrix:

I(a): "cursor" starts at upper left-hand entry.

I(b): if necessary, move cursor to right until you reach a nonzero column.

I(c): if cursor entry = 0, exchange cursor row with SWAP 1st row below it having nonzero entry in cursor column.

I(d): eliminate entries below the cursor (in cursor column) REPLACE by adding multiples of cursor row to rows below.

I(e): move the cursor down & to the right, hide all rows above it & columns to the left of it, go back to I(b).

PART II REF to RREF:

II (a): "Cursor" starts at right most leading entry first non-zero entry in a row

II (b): divide cursor row by cursor entry
SCALE

II (c): eliminate entries above cursor (by the "replace" operation)
REPLACE

II (d): move cursor left & up to the next leading entry,
go back to II (b).

Again, stop when you exit the matrix.