Lecture 20 : Rank + Nullity

Let A be an $m \times n$ matrix. In particular, A has n columns. Recall that the pivot columns of A furnish a basis of the column space (i.e. span of its columns) $\text{Col}(A) \subseteq \mathbb{R}^m$; while the null space $\text{Nul}(A) \subseteq \mathbb{R}^n$ (comprising solutions of $A\vec{x} = \vec{0}$) has a basis whose elements correspond to the non-pivot variables (since they parametrise the general solution).

Definition: (a) The rank of A is

$$\text{rank}(A) := \dim(\text{Col}(A)) = \#\text{ of pivot columns}$$
$$= \#\text{ of leading 1's in rref}(A).$$

(b) The nullity of A is

$$\text{nullity}(A) := \dim(\text{Nul}(A)) = \#\text{ of non-pivot variables}$$
$$= \#\text{ of non-pivot columns}.$$

Theorem 1: $\text{rank}(A) + \text{nullity}(A) = n.$

Proof: $(\#\text{ of pivot columns}) + (\#\text{ of non-pivot columns})$

$= \#$ of columns! $\square$

Ex 1 / Suppose that a homogeneous linear system $A\vec{x}=\vec{0}$ of 8 equations in 15 unknowns has (exactly) 10 independent solutions. If we want to turn this into a <u>consistent inhomogeneous</u> system $A\vec{x}=\vec{b}$, how many parameters govern the allowable choices for $\vec{b}$? Of course, $\vec{b}$ must lie in $\mathrm{Col}(A)$, whose dimension $= \mathrm{rank}(A) = 15 - \mathrm{nullity}(A)$ $= 15 - 10 = \underline{\underline{5}}.$ //

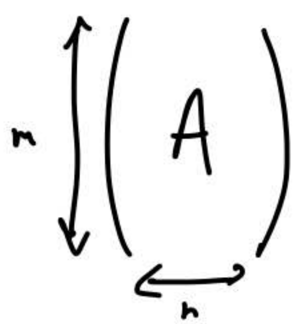
<u>Q</u>: What is the largest possible rank of an $m \times n$ matrix?

Well, $\mathrm{rank}(A) = \#$ of pivot columns
$= \#$ of leading 1's in $\mathrm{rref}(A)$
$= \#$ of nonzero rows in $\mathrm{rref}(A)$

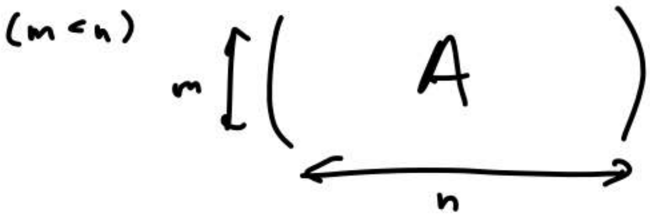
So <u>$\mathrm{rank}(A) \le \min\{m,n\}$</u>. When "=", we say that A has "maximal rank".

<u>Q</u>: What does maximal rank "look like"? 3 poss. cases:

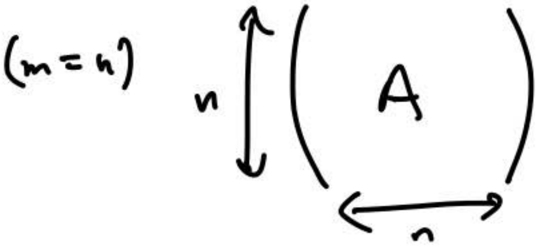
$(m > n)$



If $\mathrm{rank}(A) = n$, then (by Thm. 1) $\mathrm{nullity}(A) = 0$. So the transformation $T: \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto A\vec{x}$ is <u>1-to-1</u>.



If $\text{rank}(A) = m$, then
$m = \dim(\text{Col}(A))$
$= \dim(\text{image}(T))$,
and so T is onto.



If $\text{rank}(A) = n$, then
$\text{nullity}(A) = 0 \underset{\text{IMT}}{\Longrightarrow} A$ invertible
$\Longrightarrow T$ is 1-1 & onto (isomorphism).

Remark: (i) For square ($n \times n$) matrices, "$\text{rank}(A) = n$" (equivalently, "$\text{nullity}(A) = 0$") is yet another condition equivalent to invertibility of A.

(ii) For non-square matrices, maximal rank is the closest you can get to invertibility. In general, when $AB = I_m$ (A $m \times n$ & B $n \times m$, $m \leq n$), A & B both must be of maximal rank. ← so, not necessarily square

Row spaces

Let A be an $m \times n$ matrix.

Definition: $\text{Row}(A) \subset \mathbb{R}^n$ is the subspace consisting of all linear combinations of rows of A. (Can also think of this as $\text{Col}(A^T)$.) Row space is not changed by elementary row operations, so $\text{Row}(A) = \text{Row}(\text{rref}(A))$.

Ex 2/ $A = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \end{pmatrix}$. Find bases & dimensions of

$Col(A)$, $Nul(A)$, $Row(A)\ (= Col(A^T))$, $Nul(A^T)$.

($Nul(A)$ ↑ dependencies on columns of A; $Nul(A^T)$ ↑ dependencies on rows of A.)

[We could flip $A \rightsquigarrow A^T$ and compute $Col(A^T)$ & $Nul(A^T)$ directly, by row-reducing A^T. But there is a way to do all four just by row-reducing A.]

(pivot columns)

$$A \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & 1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = rref(A)$$

So a basis for $Col(A) = 1^{st} + 2^{nd}$ columns of $A = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$

$$\Longrightarrow rank(A) = 2 = \dim(Col(A)).$$

and a basis for $Nul(A) = \left(\text{solutions to } \begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \text{ with } \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(free variables: x_3, x_4, x_5)

$$\Longrightarrow nullity(A) = 3 = \dim(Nul(A)).$$

A basis for $Row(A) = Row(rref(A))$ is $\{(1\ 0\ 1\ 1\ 0), (0\ 1\ 1\ 0\ 0)\}$

$$\Longrightarrow \dim(Row(A)) = 2$$

while for $Nul(A^T)$, the basis is just $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ since -1^{st} row of A $+ 3^{rd}$ row of A $= \vec{0}$

$$\Longrightarrow \dim(Nul(A^T)) = 1.$$

Notice that $2+3=5$ is "rank + nullity" for A, while

$2+1=3$ is "rank + nullity" for A^T. //

Remark: In general, you could find a basis for $Nul(A^T)$ by using augmented matrices:

$$[A \mid I_3] \xrightarrow[\text{operations}]{\text{row-}} \left[\begin{array}{ccccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

these keep a record of row operations:

row1(rref) = −row1(A) + 2 row2(A)

row2(rref) = row1(A) − row2(A)

0 = row3(rref) = −row1(A) + row3(A)

The rows in the 3×3 to the right of the bar following rows of all 0's in rref(A), yield a basis for $Nul(A^T)$. (this is why) → dependency //

Theorem 2: $\dim(Row(A)) = rank(A)$ $(= \dim(Col(A)))$

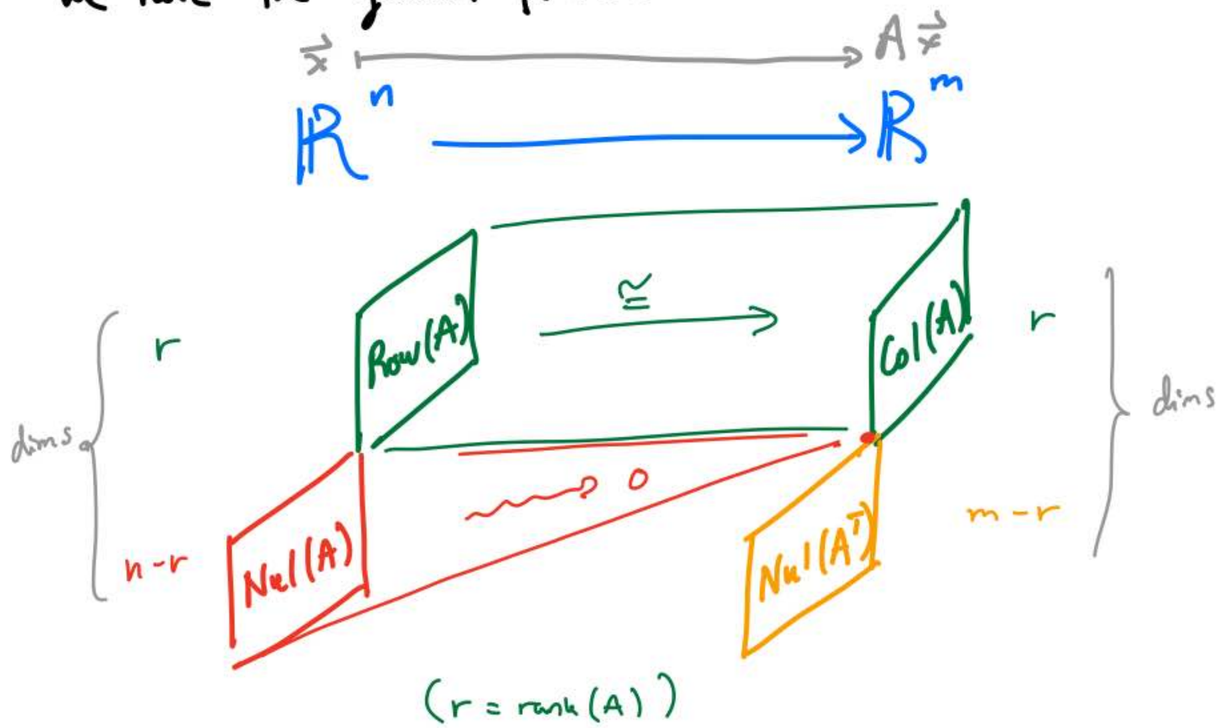
Proof: The nonzero rows of rref(A) are a basis for Row(A). They clearly span it, by reversing the elementary row operations. They are independent b/c the 'leading 1' in each nonzero row occurs in a different column, as the sole nonzero entry in that column. So $\dim(Row(A))$ = # nonzero rows = # leading 1s = rank(A). □

Another way of saying this is that A & A^T always have the same rank.

Ex 3/ Consider a homogeneous system of 12 linear equations in 8 unknowns with 2 independent solutions (& no more). Can we reduce the # of equations?

Represent as $A\vec{x} = \vec{0}$, A 12×8; a basis for Row(A) corresponds to a "minimal" set of equations. The number thereof = rank(A) = 8 − nullity(A) = 8 − 2 = 6. So indeed, we can eliminate half of the original equations. //

To summarize our discussion of the 4 subspaces above, we have the general picture



and one <u>can</u> in fact show that Row(A) is mapped onto Col(A) in 1-to-1 fashion.

Linear Transformation version

Let $T: V \to W$ be a linear transformation between two (finite-dimensional) vector spaces. Identifying V & W with $\mathbb{R}^n$ & $\mathbb{R}^m$, so that T has an $m \times n$ matrix A, allows us to reinterpret Theorem 1:

Theorem 1′: $\underbrace{\dim(\mathrm{Im}(T))}_{\text{rank}(A)} + \underbrace{\dim(\mathrm{Ker}(T))}_{\text{nullity}(A)} = \dim(V)$.

rank(A), since Im(T) identifies w/ Col(A) — nullity(A)

Naturally, we call $\dim(\mathrm{Im}(T))$ the <u>rank</u> of T and $\dim(\mathrm{Ker}(T))$ the <u>nullity</u> of T.

Ex 4/ If $T: V \xrightarrow{\cong} W$ is an isomorphism, then $\mathrm{Ker}(T) = \{0\}$ & $\mathrm{Im}(T) = W$. But then Thm. 1′ says $\dim W = \dim V$. //

Ex 5/ If $T = \frac{d}{dt} : \mathbb{P}_3 \to \mathbb{P}_3$ (polynomials of degree ≤ 3), nullity = 1 & rank = 3 ($\mathrm{Ker}(T)$ = constants) ($\mathrm{Im}(T) = \mathbb{P}_2$). Indeed $1+3 = 4 = \dim \mathbb{P}_3$. //

Ex 6/ If $T = \left(\frac{d}{dt}\right)^2 + 1 : W \to W$, where W is the 4-dim'l space of functions spanned by $\cos(t), \sin(t), t\cos(t), t\sin(t)$, then we found we could solve $Tf = \cos(t)$ in W. The general solution of $Tf = 0$ was $\alpha\cos(t) + \beta\sin(t)$ ($\Rightarrow \dim(\mathrm{Ker}(T)) = 2$). Can we solve $Tf = g$ in W for <u>any</u> $g \in W$? <u>NO</u>. Only for those g in $\mathrm{Im}(T)$, which has dimension $\dim(\mathrm{Im}(T)) = 4 - \dim(\mathrm{Ker}(T)) = 4 - 2 = 2$. //