Lecture 21: Change of Basis

A few lectures back we discussed writing vectors $\vec{v} \in V$ with respect to a basis $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for V:

$$[\vec{x}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \longleftrightarrow \vec{x} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n.$$

When $V = \mathbb{R}^n$, we had the change-of-basis matrix

$$P_B = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix}$$

and the relation

$$P_B [\vec{x}]_B = \vec{x}.$$

Ex 1/ $V = \mathbb{R}^2$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Find coordinates of $\vec{x}$ with respect to the rotated axes $B = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}$

Write $P_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so $P_B^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ (why?).

Then $[\vec{x}]_B = P_B^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}$.

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Ex 2/ Let V be some 2-dimensional vector space with bases

$$B = \{\vec{b}_1, \vec{b}_2\} \quad \text{and} \quad C = \{\vec{c}_1, \vec{c}_2\}.$$

Suppose we're given $\vec{b}_1 = -\vec{c}_1 + 4\vec{c}_2$ & $\vec{b}_2 = 3\vec{c}_1 - 6\vec{c}_2$.
If $\vec{x} = 2\vec{b}_1 + 3\vec{b}_2$, what is $[\vec{x}]_C$?

Write $\vec{x} = 2(-\vec{c}_1 + 4\vec{c}_2) + 3(3\vec{c}_1 - 6\vec{c}_2)$

$$= (-1\cdot2 + 3\cdot3)\vec{c}_1 + (4\cdot2 - 6\cdot3)\vec{c}_2 = 7\vec{c}_1 - 10\vec{c}_2.$$

Notice that this also can be rewritten as ...

$$[\vec{x}]_C = \begin{pmatrix} -1 & 3 \\ 4 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{pmatrix}\begin{pmatrix} [\vec{x}]_B \end{pmatrix} =: \underset{C \leftarrow B}{P}\, [\vec{x}]_B$$

More generally, if $C = \{\vec{c}_1, \ldots, \vec{c}_n\}$ and $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$
are bases for V, set

$$\underset{C \leftarrow B}{P} := \begin{pmatrix} [\vec{b}_1]_C & \cdots & [\vec{b}_n]_C \end{pmatrix}$$

So that if $[\vec{x}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, then

$$\underset{C \leftarrow B}{P}[\vec{x}]_B = \alpha_1 [\vec{b}_1]_C + \cdots + \alpha_n [\vec{b}_n]_C = [\alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n]_C = [\vec{x}]_C.$$

Properties: Since $\underset{C \leftarrow B}{P}[\vec{x}]_B = [\vec{x}]_C$, $\underset{C \leftarrow B}{P^{-1}}[\vec{x}]_C = [\vec{x}]_B = \underset{B \leftarrow C}{P}[\vec{x}]_C$
for all $\vec{x}$, and so

(i) $\underset{C \leftarrow B}{P^{-1}} = \underset{B \leftarrow C}{P}$.

Given another basis A, we have also

(ii) $\underset{C \leftarrow B}{P} \cdot \underset{B \leftarrow A}{P} = \underset{C \leftarrow A}{P}$.

Ex 3/ Rewrite $f(t) = 5 - 6t + 3t^2 + t^3$ in powers of $(t-1)$.

In P_3, write $B = \{1, t, t^2, t^3\}$ and $C = \{1, t-1, (t-1)^2, (t-1)^3\}$,

so that $[f]_B = \begin{pmatrix} 5 \\ -6 \\ 3 \\ 1 \end{pmatrix}$. Now $\underset{B \leftarrow C}{P} = \begin{pmatrix} \uparrow & & \uparrow \\ [\vec{c_1}]_B & \cdots & [\vec{c_4}]_B \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\underset{\substack{\text{take inverse} \\ \text{using RREF}}}{\Longrightarrow} \quad \underset{C \leftarrow B}{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and thus $[f]_C = \underset{C \leftarrow B}{P} [f]_B = \begin{pmatrix} 3 \\ 3 \\ 6 \\ 1 \end{pmatrix}$.

That is, $f(t) = 3 + 3(t-1) + 6(t-1)^2 + (t-1)^3$. $\quad /\!/$

Now let $V = \mathbb{R}^n$, $\mathcal{E} = \{\vec{e_1}, \dots, \vec{e_n}\}$ (standard basis), and

B & C other bases.

Properties: Since $P_B [\vec{x}]_B = \vec{x} = [\vec{x}]_{\mathcal{E}}$, really

• $P_B = \underset{\mathcal{E} \leftarrow B}{P}$, and $P_B^{-1} = \underset{B \leftarrow \mathcal{E}}{P}$.

Combining this with (ii) above,

• $P_C^{-1} P_B = \underset{C \leftarrow \mathcal{E}}{P} \cdot \underset{\mathcal{E} \leftarrow B}{P} = \underset{C \leftarrow B}{P}$.

Think: P_B sends $[\vec{x}]_B$ to $\vec{x}$, then P_C^{-1} sends $\vec{x}$ to $[\vec{x}]_C$.

Ex 4/ In $\mathbb{R}^2$, find the change-of-basis matrix from

$$C = \left\{ \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\} \quad \text{to} \quad B = \left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}.$$

$P_B = \begin{pmatrix} 1 & -2 \\ -5 & 2 \end{pmatrix} \Rightarrow P_B^{-1} = \frac{1}{1 \cdot 2 - (-2)(-5)} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} = -\frac{1}{8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix}$

$\Rightarrow \underset{B \leftarrow C}{P} = P_B^{-1} P_C = \frac{-1}{8} \begin{pmatrix} 2 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -3 & \frac{1}{2} \\ -5 & \frac{7}{4} \end{pmatrix}$ $\quad /\!/$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\vec{c_1}]_B \quad [\vec{c_2}]_B$

Here is another way to proceed. Consider that we want
to find the coordinate vectors of $\vec{c_1}, \vec{c_2}$ w.r.t B. That is,
we want to solve

$$P_B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{c_1} \quad , \quad P_B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \vec{c_2} ,$$

for which one would ordinarily row-reduce $\left[P_B \mid \vec{c_1} \right], \left[P_B \mid \vec{c_2} \right]$.
An even better way to think of this is : adjoining both
$\vec{c_1}$ & $\vec{c_2}$ to P_B , we get

$$\left[P_B \mid P_C \right] = \left[\vec{b_1}, \vec{b_2} \mid \vec{c_1}, \vec{c_2} \right] \xrightarrow[RREF]{} \left[\mathbb{I}_2 \mid M \right].$$

the desired $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

To see what M is, notice that the row-reduction
has to multiply P_B by P_B^{-1} to get I_2. So

$$M = P_B^{-1} P_C = \underset{B \leftarrow e}{P} .$$

Ex 4
(cont'd)

$$\begin{bmatrix} 1 & -2 \mid 7 & -3 \\ -5 & 2 \mid 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \mid 7 & -3 \\ 0 & -8 \mid 40 & -14 \end{bmatrix}$$

$\vec{b_1} \quad \vec{b_2} \quad \vec{c_1} \quad \vec{c_2}$

$$\rightarrow \begin{bmatrix} 1 & -2 \mid 7 & -3 \\ 0 & 1 \mid -5 & 7/4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 \mid -3 & 1/2 \\ 0 & 1 \mid -5 & 7/4 \end{bmatrix}$$