# Lecture 26: Matrix of a Linear Transformation

Just as it is useful to write vectors $\vec{v} \in V$ in terms of their coordinates with respect to a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ of $V$, viz.

$$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{if} \quad \vec{v} = a_1 \vec{b}_1 + \dots + a_n \vec{b}_n \quad ,$$

it is also computationally convenient to express linear transformations in terms of their matrices of coefficients with respect to two bases (one for the domain & one for the codomain). [What does this have to do with "eigenstuff"? You'll see!]

Namely, given a linear transformation

$$T : V \to W$$

with bases $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for $V$ & $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_m\}$ for $W$, we write

$$_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} \quad \text{if} \quad T(\vec{b}_j) = \alpha_{1j} \vec{c}_1 + \dots + \alpha_{mj} \vec{c}_m$$

(for each $j = 1, \dots, n$).

Since $[T(\vec{b}_j)]_{\mathcal{C}} = \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix}$, we can rewrite this matrix as

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{pmatrix} \uparrow & & \uparrow \\ [T(\vec{b}_1)]_{\mathcal{C}} & \cdots & [T(\vec{b}_n)]_{\mathcal{C}} \\ \downarrow & & \downarrow \end{pmatrix}. $$

In this form clearly

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}}\,[\vec{b}_j]_{\mathcal{B}} = \begin{pmatrix} \uparrow & & \uparrow \\ [T(\vec{b}_1)]_{\mathcal{C}} & \cdots & [T(\vec{b}_n)]_{\mathcal{C}} \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \uparrow \\ [T(\vec{b}_j)]_{\mathcal{C}} \\ \downarrow \end{pmatrix}, $$

$j^{\text{th}}$ place

and so by extending linearly to combinations of the $\{\vec{b}_j\}$,

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}}\,[\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{C}}. $$

Ex 1/ Let $V = \text{span}\underbrace{\{\cos(t), \sin(t), t\cos(t), t\sin(t)\}}_{\mathcal{B}} \subset C^\infty(\mathbb{R})$, and

$$ T : V \to \mathbb{P}_4 \ (= \text{span}\{1, t, t^2, t^3, t^4\}) $$

to take the $4^{\text{th}}$-order Taylor polynomial about $0$:

so

$$T(\underbrace{\cos(t)}_{\vec{b}_1}) = 1 - \frac{t^2}{2} + \frac{t^4}{24}$$

$$T(\underbrace{\sin(t)}_{\vec{b}_2}) = t - \frac{t^3}{6}$$

$$T(\underbrace{t\cos(t)}_{\vec{b}_3}) = t - \frac{t^3}{2}$$

$$T(\underbrace{t\sin(t)}_{\vec{b}_4}) = t^2 - \frac{t^4}{6}$$

$$_C[T]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{6} & -\frac{1}{2} & 0 \\ \frac{1}{24} & 0 & 0 & -\frac{1}{6} \end{pmatrix}$$

**Ex 2/** Here is another example which actually does something.

Let $V = \mathbb{P}_2(x,y)$ be the vector space of polynomials of degree $\leq 2$ in $x$ and $y$

$B = \{1, x, y, x^2, xy, y^2\}$ basis for $V$

$W = \mathbb{R}^5$ with the standard basis $\mathcal{E}$.

Now choose 5 points $A = (x_A, y_A)$, $B = (x_B, y_B)$, $C = (x_C, y_C)$, $D = (x_D, y_D)$, $E = (x_E, y_E)$ in the plane, and define

$T : V \longrightarrow W$ by evaluation of functions at these 5 points.

$$p(x,y) \longmapsto \begin{pmatrix} p(x_A, y_A) \\ p(x_B, y_B) \\ p(x_C, y_C) \\ p(x_D, y_D) \\ p(x_E, y_E) \end{pmatrix}$$

The matrix of this transformation is

$$
{}_{\mathcal{E}}[T]_{\mathcal{B}} = \begin{pmatrix} 1 & x_A & y_A & x_A^2 & x_A y_A & y_A^2 \\ 1 & x_B & y_B & x_B^2 & x_B y_B & y_B^2 \\ 1 & x_C & y_C & x_C^2 & x_C y_C & y_C^2 \\ 1 & x_D & y_D & x_D^2 & x_D y_D & y_D^2 \\ 1 & x_E & y_E & x_E^2 & x_E y_E & y_E^2 \end{pmatrix}.
$$

As long as no 4 of the points are collinear, one can check* this matrix has rank 5, hence nullity 1. Since the null space of the matrix is the kernel $\ker(T)$ written in the basis $\mathcal{B}$, that means that $\ker(T)$ is spanned by a single polynomial $f$. ($\dagger$)

What does this mean? For a polynomial $f$ to satisfy $T(f) = \vec{0}$ is to say that $f$ evaluates to 0 at all 5 points. Equivalently, the conic (ellipse / hyperbola / whatever) defined by $f(x,y) = 0$ passes through $A, B, C, D, E$. We can thus interpret ($\dagger$) as saying that through most configurations of 5 points there passes exactly one conic! //

A special case of the above is

$$T : V \to V$$

where $\mathcal{B} = \mathcal{C}$. We will write $[T]_{\mathcal{B}}$ instead of ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ in this case.

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* That goes beyond our scope here, but at least it's clear that $\mathrm{rk} \leq 5$ so nullity $\geq 1$.

Ex 3/ $T = \frac{d}{dt}$ on $V = \mathbb{P}_3$ = polynomials of degree $\le 3$;

that is,

$$\frac{d}{dt} : \mathbb{P}_3 \to \mathbb{P}_3 .$$

It sends $1 \mapsto 0$, $t \mapsto 1$, $t^2 \mapsto 2t$, $t^3 \mapsto 3t^2$. So

in terms of $\mathcal{B} = \{1, t, t^2, t^3\}$, we have

$$\left[\frac{d}{dt}\right]_{\mathcal{B}} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \left[\frac{d}{dt}1\right]_{\mathcal{B}} & \left[\frac{d}{dt}t\right]_{\mathcal{B}} & \left[\frac{d}{dt}t^2\right]_{\mathcal{B}} & \left[\frac{d}{dt}t^3\right]_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad //$$

Now let's specialize further: to $V = \mathbb{R}^n$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the transformation given by $\vec{v} \longmapsto A\vec{v}$ for some $n \times n$ matrix $A$.

Clearly, $A = [T]_{\mathcal{E}}$ ($\mathcal{E}$ = standard basis). How do we find $[T]_{\mathcal{B}}$ for some other basis $\mathcal{B}$ of $\mathbb{R}^n$?

(Remember, we must have $[T]_{\mathcal{B}} [\vec{v}]_{\mathcal{B}} = [T(\vec{v})]_{\mathcal{B}}$.)

**Theorem:** $[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$.

**Proof:** $P_{\mathcal{B}}^{-1} A \underbrace{P_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}} = P_{\mathcal{B}}^{-1} A \underbrace{\vec{v}} = P_{\mathcal{B}}^{-1} T(\vec{v})$

$= [T(\vec{v})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\vec{v}]_{\mathcal{B}}$. $\square$

**Ex 4/** Find the $\mathcal{B}$-matrix of the transformation

$$\vec{x} \stackrel{T}{\longmapsto} A\vec{x},$$

where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ and $A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$.

$P_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \to P_{\mathcal{B}}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

So $[T]_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$. //

**So why is this useful?**

**Ex 5/** Find a matrix for rotating $\mathbb{R}^3$ $90°$ about (the axis spanned by) $\vec{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, given that $\vec{b}_1$, $\vec{b}_2 = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$, and $\vec{b}_3 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$ are perpendicular (their dot products $\vec{b}_1 \cdot \vec{b}_2$, $\vec{b}_1 \cdot \vec{b}_3$, $\vec{b}_2 \cdot \vec{b}_3$ are all $0$), and $\vec{b}_2$ & $\vec{b}_3$ are the same length.

The rotation must have $T(\vec{b}_1) = \vec{b}_1$, $T(\vec{b}_2) = \vec{b}_3$, $T(\vec{b}_3) = -\vec{b}_2$.

So $[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, and

$$A = P_{\mathcal{B}} [T]_{\mathcal{B}} P_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$$

$= \frac{1}{9}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}$ — not a matrix you would have guessed! //

Why is this relevant to eigenvectors & eigenvalues?

If $\mathcal{B}$ is an eigenbasis, say in $\mathbb{R}^3$, then

$$T(\vec{b}_1) = \lambda_1 \vec{b}_1, \quad T(\vec{b}_2) = \lambda_2 \vec{b}_2, \quad T(\vec{b}_3) = \lambda_3 \vec{b}_3 \implies$$

$$[T]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \text{ is } \underline{\underline{\text{diagonal}}} \implies$$

$P_{\mathcal{B}}^{-1} A P_{\mathcal{B}}$ is diagonal (faster than our earlier approach!).

To reiterate:

$$\boxed{[T]_{\mathcal{B}} \text{ is diagonal} \iff \mathcal{B} \text{ is an eigenbasis for } T.}$$