# Lecture 28: Discrete Dynamical Systems

Let $A$ be a diagonalizable $n\times n$ matrix, and $\mathcal{B}=\{\vec{v}_1,\dots,\vec{v}_n\}$ the corresponding eigenbasis, with $A\vec{v}_i=\lambda_i\vec{v}_i$, $i=1,\dots,n$.

We are interested in the "dynamical system"

$$(*) \qquad \vec{x}_{k+1} = A\vec{x}_k$$

with initial state vector $\vec{x}_0$. If $\vec{x}_0=c_1\vec{v}_1+\dots+c_n\vec{v}_n$, applying $A$ $k$ times yields

$$\vec{x}_k = A^k\vec{x}_0 = A^k(c_1\vec{v}_1+\dots+c_n\vec{v}_n) = c_1A^k\vec{v}_1+\dots+c_nA^k\vec{v}_n$$
$$= c_1\lambda_1^k\vec{v}_1+\dots+c_n\lambda_n^k\vec{v}_n .$$

A more conceptual approach would be to use a change of basis. Writing $T:\mathbb{R}^n\to\mathbb{R}^n$ for the transformation $T(\vec{x})=A\vec{x}$,

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}AP_{\mathcal{B}} = D = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

and $(*)$ becomes

$$(**) \qquad [\vec{x}_{k+1}]_{\mathcal{B}} = [T(\vec{x}_k)]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}_k]_{\mathcal{B}} .$$

Setting

$$\vec{y}_k := [\vec{x}_k]_{\mathcal{B}} = \begin{pmatrix} y_1(k)\\ \vdots \\ y_n(k)\end{pmatrix},$$

$(**)$ translates into

$$\begin{pmatrix} y_1(k+1) \\ \vdots \\ y_n(k+1) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1(k) \\ \vdots \\ y_n(k) \end{pmatrix}$$

or simply

$$(***) \qquad y_j(k+1) = \lambda_j y_j(k) \qquad (j = 1, \dots, n).$$

That is, we have "decoupled" the dynamical system: each equation involves only one $y_j$.

**Ex 1/** $A = 2\times 2$ matrix w/ eigenvalues $\lambda_1 = 3$, $\lambda_2 = \frac{1}{3}$ and eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\vec{x}_0 = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$.

(a) Compute $\vec{x}_1$.

$$\vec{x}_1 = A\vec{x}_0 = P_B D P_B^{-1} \vec{x}_0 = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix} \cdot \underbrace{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 1 \end{pmatrix}}_{=\begin{pmatrix} 5 \\ -4 \end{pmatrix} \text{ : that is, } \vec{x}_0 = 5\vec{v}_1 - 4\vec{v}_2 \text{ (or } c_1 = 5,\ c_2 = -4)}$$

$$= \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 3\cdot 5 \\ \frac{1}{3}\cdot -4 \end{pmatrix} = 15\vec{v}_1 - \frac{4}{3}\vec{v}_2 = \begin{pmatrix} 15 \\ 15 \end{pmatrix} - \begin{pmatrix} -4/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 49/3 \\ 41/3 \end{pmatrix}.$$

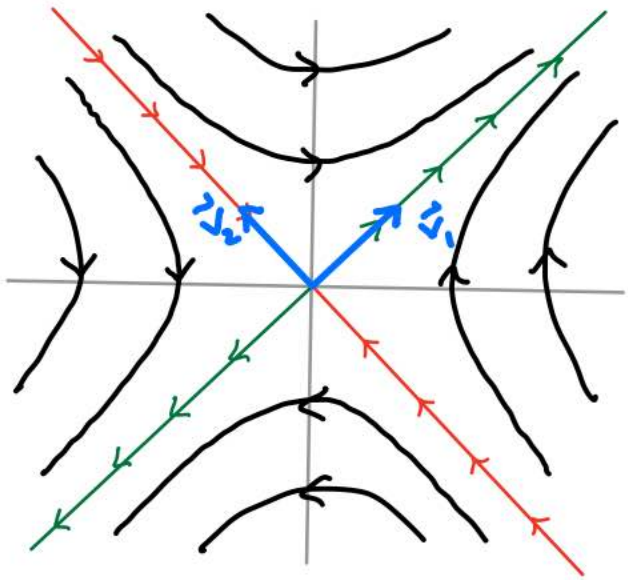
(b) Find a formula for $\vec{x}_k$ involving $k$ and the eigenvectors $\vec{v}_1$ & $\vec{v}_2$.

$$\vec{x}_k = A^k \vec{x}_0 = (P_B D P_B^{-1})^k \vec{x}_0 = P_B D^k \underbrace{P_B^{-1} \vec{x}_0}_{\begin{pmatrix} 5 \\ -4 \end{pmatrix}}$$

$$= \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & (\frac{1}{3})^k \end{pmatrix} \begin{pmatrix} 5 \\ -4 \end{pmatrix} = 5\cdot 3^k \vec{v}_1 - 4\cdot\left(\frac{1}{3}\right)^k \vec{v}_2$$

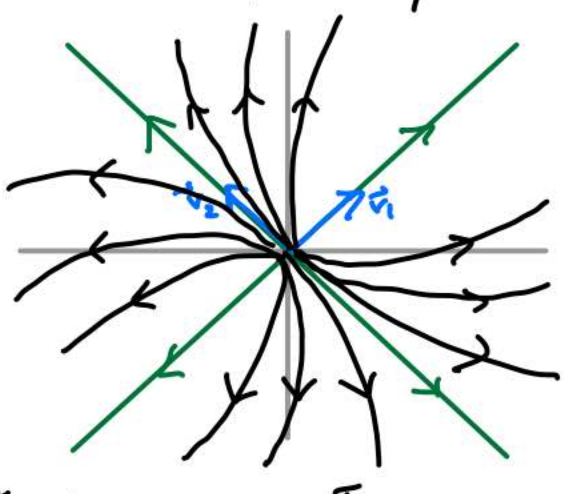
$$= \begin{pmatrix} 5\cdot 3^k + 4(\frac{1}{3})^k \\ 5\cdot 3^k - 4(\frac{1}{3})^k \end{pmatrix}.$$

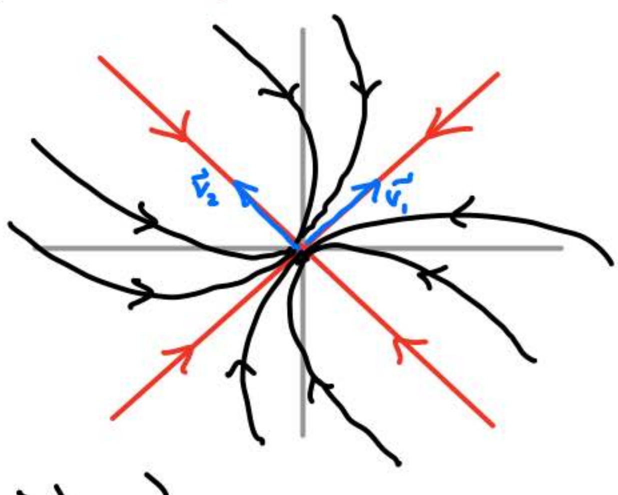
(c) Sketch a "phase portrait" for the system. (This is a fancy way of saying "sketch the trajectories of solutions.")




$\vec{x}_k$ "increases in the $\pm\vec{v}_1$-direction" (as $k$ increases) since $\lambda_1 > 1$, and "decreases in the $\pm\vec{v}_2$-direction" since $0 < \lambda_2 < 1$. In the long term, $\vec{x}_k$ is very close to being a very large multiple of $\pm\vec{v}_1$, since $\left(\frac{1}{3}\right)^k \to 0$. //

The picture in (c) is said to have a saddle point at the origin; this occurs whenever $\lambda_1 > 1 > \lambda_2 > 0$. If both $\lambda_1, \lambda_2 > 1$, the origin is a repeller; while if $1 > \lambda_1, \lambda_2 > 0$, the origin is an attractor:




(In these pictures I assume $\lambda_1 > \lambda_2$.)

Ex 2/ On a campground, the grizzly and human populations at time $k$ are described by

$$\vec{x}_k = \begin{pmatrix} G_k \\ H_k \end{pmatrix} \quad \text{(measured in hundreds)}$$

and they are subject to the equations

$$\begin{cases} G_{k+1} = 0.5 \cdot G_k + 0.4 \cdot H_k \\ H_{k+1} = -p \cdot G_k + 1.1 \cdot H_k \, . \end{cases}$$

What happens when the predation parameter $p = \begin{cases} \text{(a) } 0.104 \\ \text{(b) } 0.2 \\ \text{(c) } 0.625 \end{cases}$ ?

(a) is worked out in the book (Example 1 in §5.6):

$$\lambda_1 = 1.02, \; \lambda_2 = 0.58 \quad \leftrightarrow \quad \vec{v}_1 = \begin{pmatrix} 10 \\ 13 \end{pmatrix}, \; \vec{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$\Rightarrow$ origin is a saddle point and system tends toward 13(00) people for every 10(00) grizzlies.

(b) $0 = \det(A - \lambda I) = (0.5 - \lambda)(1.1 - \lambda) + 0.4(0.2)$

$= \lambda^2 - 1.6\lambda + 0.63 \quad \Rightarrow \quad \lambda = 0.8 \pm \sqrt{0.64 - 0.63} = \begin{cases} 0.9 \\ 0.7 \end{cases}$

$\Longrightarrow$ origin is attractor and everything dies off.

(c) $0 = \det(A - \lambda I) = \lambda^2 - 1.6\lambda + 0.8$

$\longrightarrow \lambda = 0.8 \pm \sqrt{0.64 - 0.8} = 0.8 \pm 0.4i$. What happens?

We have an eigenvalue $a - bi$ with $a = 0.8$, $b = 0.4$

and eigenvector $\vec{v}$:

$$E_{0.8-0.4i} = \mathrm{Nul}\begin{pmatrix} -0.3+0.4i & 0.4 \\ -0.625 & 0.3+0.4i \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} -\frac{3}{4}+i & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ \frac{3}{4}-i \end{pmatrix}}_{\vec{v}} \right\}.$$

Separating this into real & imaginary parts

$$\vec{v} = \begin{pmatrix} 1 \\ \frac{3}{4}-i \end{pmatrix} = \begin{pmatrix} 1 \\ 3/4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix} =: \vec{u} + i\vec{w} \implies P = \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 3/4 & -1 \end{pmatrix} \\ \;\;\vec{u}\quad\;\vec{w} \end{array},$$

and by lecture 27

$$A = P\begin{pmatrix} a & -b \\ b & a \end{pmatrix}P^{-1} = P \cdot rR_\theta \cdot P^{-1},$$

where $r = \sqrt{a^2+b^2} = \sqrt{0.8} < 1$ and $\theta = \arctan(1/2)$.

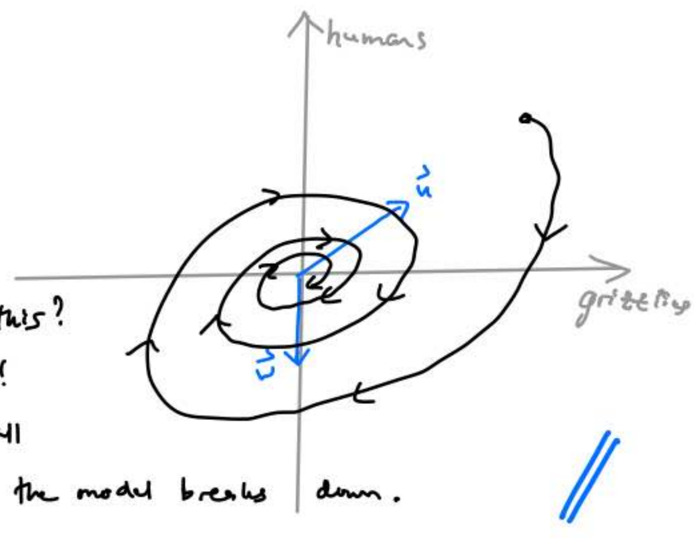
So if $\vec{x}_0 = c_1\vec{u} + c_2\vec{w}$, $P^{-1}\vec{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and

$$\vec{x}_k = A^k\vec{x}_0 = P(rR_\theta)^k P^{-1}\vec{x}_0 = r^k P R_{k\theta}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 r^k \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & \vec{w} \\ \downarrow & \downarrow \end{pmatrix}\begin{pmatrix} \cos(k\theta) \\ \sin(k\theta) \end{pmatrix} = c_1(0.8)^{k/2}\left\{(\cos(k\theta))\vec{u} + (\sin(k\theta))\vec{w}\right\},$$

simplify: assume $c_2 = 0$

which gives spiralling trajectories with the origin as attractor:




[Note that the spiralling is from $\vec{u}$ towards $\vec{w}$ as the angle $\theta$ increases.]

What is the "physical" meaning of this? You can't have negative populations! It means that the grizzlies eat all the humans, and then of course the model breaks down.

Let's turn, in anticipation of Lecture 29, to a continuous dynamical system:

$$\frac{dx_1}{dt} = x_1(t) + 2x_2(t)$$
$$\frac{dx_2}{dt} = -x_1(t) + 4x_2(t).$$

The corresponding matrix equation is

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \left(A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix},\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$
$$= P_{\mathcal{B}} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} P_{\mathcal{B}}^{-1} \vec{x} \qquad \left(P_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

If $\vec{x}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 = P_{\mathcal{B}}\vec{c}(t)$, then also

$$\frac{d\vec{x}}{dt} = P_{\mathcal{B}} \frac{d\vec{c}}{dt} \qquad \text{(entries of } P_{\mathcal{B}} \text{ being constant)}$$

and so the system becomes

$$P_{\mathcal{B}} \frac{d\vec{c}}{dt} = P_{\mathcal{B}} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{c}(t)$$

or $$\frac{d\vec{c}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{c}(t).$$

This is just the two equations

$$\begin{cases} \dfrac{dc_1}{dt}(t) = 2c_1(t) \\[2ex] \dfrac{dc_2}{dt}(t) = 3c_2(t) \end{cases}$$

(i.e. we have decoupled the system), which have solutions

$$\begin{cases} c_1(t) = c_1(0)\, e^{2t} \\ c_2(t) = c_2(0)\, e^{3t} \end{cases}$$

$$\Rightarrow \vec{x}(t) = c_1(0) e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2(0)\, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So the only obvious difference with the discrete case is " $e^{\lambda t}$ " vs. " $\lambda^t$ ". This is in fact a big difference: for $0 < \lambda < 1$, $e^{\lambda t}$ is an increasing function of $t$, while $\lambda^t$ is a decreasing one! To be continued...