Lecture 29: Systems of Differential Equations

In this lecture we continue our investigation of "continuous dynamical systems" from the end of Lecture 28 — that is, how to solve systems of the form

(1) $$\vec{x}\,'(t) = A\vec{x}(t)$$

with A an $n\times n$ matrix.

Real-diagonalizable case

Suppose that $A = P_{\mathcal{B}} D P_{\mathcal{B}}^{-1}$, with $P_{\mathcal{B}} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. Then

$$\vec{x}\,'(t) = P_{\mathcal{B}} D P_{\mathcal{B}}^{-1}\vec{x}(t) \implies P_{\mathcal{B}}^{-1}\vec{x}\,'(t) = D P_{\mathcal{B}}^{-1}\vec{x}(t)$$

(and setting $\vec{y}(t) = P_{\mathcal{B}}^{-1}\vec{x}(t)$, $\vec{c} = \vec{y}(0) = P_{\mathcal{B}}^{-1}\vec{x}(0)$)

$$\implies \vec{y}\,'(t) = D\vec{y}(t)$$

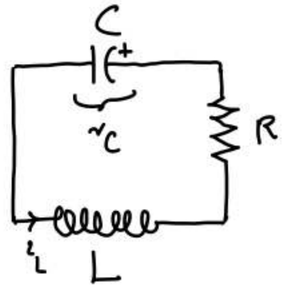
$$\implies y_j'(t) = \lambda_j y_j(t) \quad (\text{for each } j = 1, \dots, n).$$

$$\implies y_j(t) = c_j e^{\lambda_j t} \quad (\quad " \quad)$$

This yields the general solution to (1) in the real-diagonalizable case:

(2) $$\vec{x}(t) = P_{\mathcal{B}}\vec{y}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t}\vec{v}_j .$$

Ex 1/ Consider the electrical circuit:



i_L = current thru inductor

v_C = charge on capacitor

L = inductance

C = capacitance

R = resistance

This satisfies the equations $\begin{cases} i_L' = -\frac{R}{L} i_L - \frac{1}{L} v_C \\ v_C' = \frac{1}{C} i_L \end{cases}$, which

we can express as $\vec{x}\,'(t) = A\vec{x}(t)$ with $A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$, $\vec{x}(t) = \begin{pmatrix} i_L \\ v_C \end{pmatrix}$.

Suppose $R = 5/2$, $L = 1 = C$; then $A = \begin{pmatrix} -5/2 & -1 \\ 1 & 0 \end{pmatrix}$, and

$\det(A - \lambda I_2) = \begin{vmatrix} -\frac{5}{2} - \lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1$ has roots $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -2$.

For $\vec{v}_1$, $\mathrm{Nul}(A + \frac{1}{2}I) = \mathrm{Nul}\begin{pmatrix} -2 & -1 \\ 1 & 1/2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\vec{v}_1}\right\}$.

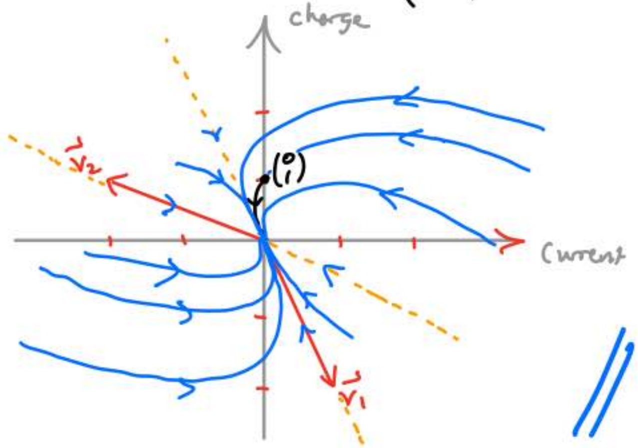
For $\vec{v}_2$, $\mathrm{Nul}(A + 2I) = \mathrm{Nul}\begin{pmatrix} -1/2 & -1 \\ 1 & 2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\vec{v}_2}\right\}$.

If $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (start w/ charge on capacitor but no current),

then $\vec{c} = P_{\mathcal{B}}^{-1}\vec{x}(0) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$.

Plugging in to (2) gives $\vec{x}(t) = -\frac{2}{3}e^{-t/2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{1}{3}e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The origin is an attractor, as it should be: the voltage is dissipating through the resistor and inductor.



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$\boxed{\text{Complex-diagonalizable case}}$

We'll work this out for 2×2 matrices only. Suppose A has an eigenvalue $\lambda = a - ib$ with corresponding eigenvector $\vec{v} = \vec{u} + i\vec{w}$ ($A, a, b, \vec{u}, \vec{w}$ all real). Write

$$c_1 \vec{v} + c_2 \overline{\vec{v}} = \vec{x}(0) \underset{\substack{\uparrow \\ \vec{x}(t)\text{ real}}}{=} \overline{\vec{x}(0)} = \overline{c_1}\,\overline{\vec{v}} + \overline{c_2}\,\vec{v}$$

$\Rightarrow c_2 = \overline{c_1}$. Set $c_1 = \frac{\alpha + i\beta}{2}$, $c_2 = \frac{\alpha - i\beta}{2}$ ($\alpha, \beta \in \mathbb{R}$).

We can again use (2) to solve $\vec{x}'(t) = A\vec{x}(t)$:

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\overline{\lambda} t}\overline{\vec{v}}$$

$$= \frac{\alpha + i\beta}{2} e^{(a-ib)t}(\vec{u} + i\vec{w}) + \frac{\alpha - i\beta}{2} e^{(a+ib)t}(\vec{u} - i\vec{w})$$

$$= \frac{e^{at}}{2}\left\{ \underbrace{\left(e^{ibt} + e^{-ibt}\right)}_{2\cos bt}(\alpha\vec{u} - \beta\vec{w}) - i\underbrace{\left(e^{ibt} - e^{-ibt}\right)}_{-2\sin bt}(\beta\vec{u} + \alpha\vec{w}) \right\}$$

$e^{iz} = \cos z + i\sin z \Longrightarrow$ (for $z \in \mathbb{R}$)

(3) $$= \alpha \cdot e^{at}\left\{(\cos bt)\vec{u} + (\sin bt)\vec{w}\right\} + \beta \cdot e^{at}\left\{(\sin bt)\vec{u} - (\cos bt)\vec{w}\right\}$$

The next example is motivated by the following question: If we change the resistance in Example 1, what happens?

It turns out that the discriminant in the quadratic formula for λ is negative ($\Rightarrow$ complex roots) if $R^2 < 4\frac{L}{C}$, i.e. if the resistance is small. The capacitor will still have to "unload," but since current through the inductor has "inertia", with very little resistance that inertia will cause the current and charge to oscillate.

Ex 2/ Decrease the "R" (in Example 1) to 1, so $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$.

Then the characteristic polynomial is $\lambda^2 + \lambda + 1$, with roots

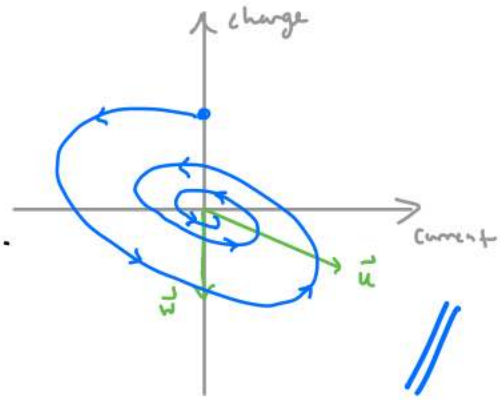
$\lambda_{\pm} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\left(\Rightarrow a = -\frac{1}{2},\ b = \frac{\sqrt{3}}{2}\right)$. We have

$$\mathrm{Nul}\left(A - (a+ib)\mathbb{I}\right) = \mathrm{Nul}\begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -1 \\ 1 & \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} = \mathrm{span}\left\{\underbrace{\begin{pmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix}}_{\vec{v}}\right\}$$

$\Rightarrow \vec{u} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 0 \\ -\sqrt{3}/2 \end{pmatrix}$. Moreover, $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{3}}\vec{v} - \frac{i}{\sqrt{3}}\overline{\vec{v}}$

$\Rightarrow \alpha = 0,\ \beta = \frac{2}{\sqrt{3}}$. By (3),

$$\vec{x}(t) = \frac{-2}{\sqrt{3}} e^{-t/2}\left\{\sin\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + \cos\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{pmatrix}\right\}$$
$$= \frac{2}{\sqrt{3}} e^{-t/2}\left\{\sin\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} -1 \\ 1/2 \end{pmatrix} + \cos\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix}\right\}.$$



Non-diagonalizable case

What happens in a case like

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where A is not diagonalizable if $a \neq 0$? (The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, but $E_1 = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ is only 1-dimensional.) Well, $e^{1t}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^t\\0\end{pmatrix}$ is a solution just like in the above cases, but how do we get a second solution? (I'll say in class.)

As one final thing to think about (to be discussed in class): can you use diagonalization to "decouple" systems of partial differential equations, like

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + 4\frac{\partial u_2}{\partial x} = 0$$
$$\frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = 0,$$

to get instead two equations of the form

$$\frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} = 0 \quad ?$$

(This is then easy to solve with given initial condition $f(x,0) = F(x)$; you simply have $f(x,t) = F(x-\lambda t)$.)